Curvature Computations of 2-manifolds in $\mathbb{R}^k$

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Abstract
In this paper, we provide simple and explicit formulas for computing Riemannian curvatures, mean curvature vectors, principal curvatures and principal directions for a 2-dimensional Riemannian manifold embedded in $\mathbb{R}^k$ with $k \geq 3$.

Key words: Riemannian curvature; mean curvature vector; principal curvatures; principal directions.

1 Introduction

The concepts of Riemannian curvatures, mean curvature vectors and principal curvatures have been developed in the field of *Riemannian Geometry*. These concepts are respectively generalizations of Gaussian curvatures, mean curvatures and principal curvatures for the classical surfaces in $\mathbb{R}^3$. It is well known that these three types curvatures for classical surfaces are extremely important notions in *Computational Geometry*, *Computer Graphics*, *Image Processing* and *Computer Added Geometric Design.* Their counterparts for 2-dimensional Riemannian manifold (abbreviated as 2-manifold) embedded in $\mathbb{R}^k$ are, as expected, equally important. Indeed, we have found that these concepts play an important role in the fields of image processing ([5, 7, 1]) and function diffusion ([2, 3]). However, the general frame of Riemannian geometry makes these curvatures difficult to calculate.

We provide simple and explicit formulas for computing Riemannian curvatures, mean curvature vectors, principal curvatures and principal directions for a 2-manifold embedded in $\mathbb{R}^k$ with $k \geq 3$. These formulas are simple compared to those found in Riemannian geometry literature ([4, 6, 8]). Individuals with little knowledge of Riemannian geometry, but who are familiar with vector computations in the Euclidean space, can easily understand and use them. Even though the starting point of the derivation of these formulas involves the use of Riemannian geometry, we have tried to minimize its use while keeping the derivation as precise as possible.

It may seem trivial for people working in the field of Riemannian geometry to derive these curvature formulas for the 2-manifold, however we have not seen these formulas presented in a simple and precise enough manner to fulfill our needs.

2 Curvature Formulas

The aim of this section is to provide readers with a quick reference for the curvature computation formulas. The detail derivation of these formulas are given in the section that follows.

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Let M be a 2-dimensional Riemannian manifold in $I\!R^k$ with a Riemannian metric defined by the scalar inner product. Let (ξ_1, ξ_2) be a local coordinate system of the 2-manifold M at the point $x \in M$. Then $x \in I\!R^k$ can be expressed as

$$x = [x_1(\xi_1, \xi_2), \cdots, x_k(\xi_1, \xi_2)]^T. \tag{2.1}$$

Let $t_i = \frac{\partial x}{\partial \xi_i}$, $t_{ij} = \frac{\partial^2 x}{\partial \xi_i \partial \xi_j}$, $g_{ij} = t_i^T t_j$, and

$$G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}, \quad Q = I - [t_1, t_2] G^{-1} [t_1, t_2]^T \in I\!R^{k \times k},$$

where

$$G^{-1} = \frac{1}{\det(G)} \begin{bmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{bmatrix}.$$

Then we have the following formulas:

Riemannian Curvature:

$$K(x) = \frac{t_{11}^T Q t_{22} - t_{12}^T Q t_{12}}{\det(G)}. \tag{2.2}$$

The Riemannian curvature is a counterpart of the Gaussian curvature of the classical surface. If $k = 3$, the Riemannian curvature coincides with the Gaussian curvature for surfaces.

Mean Curvature Vector:

$$H(x) = \frac{Q(g_{22} t_{11} + g_{11} t_{22} - 2 g_{12} t_{12})}{2 \det(G)}. \tag{2.3}$$

The mean curvature vector is a vector in the normal space. If $k = 3$, the mean curvature vector is in the normal direction, and its length is the classical mean curvature of the surface.

Principal Curvatures and Principal directions:

To obtain formulas for the principal curvatures and the principal directions, we first introduce an auxiliary result: *Let $A = (a_{ij})_{i,j=1}^2 \in I\!R^{2\times 2}$ be a symmetric matrix. Then the eigenvalues of A are*

$$\lambda_\pm = \frac{a_{11} + a_{22} \pm \sqrt{(a_{11} - a_{22})^2 + 4a_{12}^2}}{2} \tag{2.4}$$

and the corresponding eigenvectors are $[cos\theta_\pm, sin\theta_\pm]^T$, where $\theta_\pm$ are given (modulo π) by

$$\theta_+ = \frac{1}{2}\text{arctan}\frac{2a_{12}}{a_{11} - a_{22}}, \quad \theta_- = \theta_+ + \frac{\pi}{2}. \tag{2.5}$$

Now we give formulas for computing the principal curvatures and the principal directions. Let $h(x) = H(x)/\|H(x)\|$,

$$A = \Lambda^{-\frac{1}{2}} K F_h K^T \Lambda^{-\frac{1}{2}} \in I\!R^{2\times 2}, \quad [u_1, u_2] = [t_1, t_2] K^T \Lambda^{-\frac{1}{2}}, \tag{2.6}$$

where $F_h = -\left(t_{ij}^T\ h(x)\right)_{ij=1}^2$, $K \in I\!R^{2\times 2}$ and $\Lambda \in I\!R^{2\times 2}$ are defined by

$$G = K^T \Lambda K, \quad K^T K = I, \quad \Lambda = \text{diag}(\lambda_1, \lambda_2) \tag{2.7}$$

and they can be computed by (2.4)–(2.5). Let A be expressed, by virtue of (2.4) and (2.5), as

$$A = P \ \mathrm{diag}(k_1, k_2) \ P^T, \quad \text{with} \quad P^T P = I. \tag{2.8}$$

Then k_1 and k_2 are the principal curvatures and v_1 and v_2, defined by

$$[v_1, v_2] := [u_1, u_2]P = [t_1, t_2]K^T \Lambda^{-\frac{1}{2}} P, \tag{2.9}$$

are the corresponding principal directions with respect to the direction vector h.

Again, the principal curvatures and the principal directions are the counterparts of the same concepts for surfaces. If $k = 3$, they are the same.

3 Derivation

In this section we derive the curvature formulas from the field of Riemannian geometry. However, we have tried to make the paper self-contained, so that readers can understand the derivation without having to consult the Riemannian geometry literature. Readers may simply skim over this section if they merely intend to use the curvature formulas.

3.1 Notations and Terminologies

Differential Manifold. A *differentiable manifold* of dimension n is a set M and a family of injective mappings $x_\alpha : U_\alpha \subset I\!R^n \to M$ of open sets U_α into M such that

(1). $\bigcup_\alpha x_\alpha(U_\alpha) = M$.

(2). For any pair α, β with $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W \neq \emptyset$, the sets $x_\alpha^{-1}(W)$ and $x_\beta^{-1}(W)$ are open in $I\!R^n$ and the mappings $x_\beta^{-1} \circ x_\alpha$ are differentiable.

The mapping x_α with $x \in x_\alpha(U_\alpha)$ is called a parameterization of M at x. In our case, we use the 2-dimensional manifold ($n = 2$). Denoting the coordinate U_α as (ξ_1, ξ_2), then the tangent space $T_x M$ at $x \in M$ is spanned by $\{\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2}\}$. The set $TM = \{(x, v);\ x \in M,\ v \in T_x M\}$ is called a tangent bundle.

Vector Field ([4], page 25). A vector field X on a differentiable manifold M is a correspondence that associates to each point $x \in M$ a vector $X(x) \in T_x M$. The field is differentiable if the mapping $X : M \to TM$ is differentiable.

Considering a parameterization $x : U \subset I\!R^2 \to M$, there exist $a_i(x)$, such that

$$X(x) = \sum_i a_i(x) \frac{\partial}{\partial \xi_i}.$$

Let $\mathcal{D}$ be the set of differentiable functions on M, and X be a vector field on M. Then X can be regarded as a mapping $X : \mathcal{D} \to \mathcal{D}$ such that

$$(Xf)(x) = \sum_i a_i(x) \frac{\partial f}{\partial \xi_i}(x). \tag{3.1}$$

It is easy to check that Xf does not depend on the choice of parameterization x. Let X and Y be differentiable vector fields on a differentiable manifold M. Then there exists a unique vector field Z such that, for all $f \in \mathcal{D}$, $Zf = (XY - YX)f$. The vector field $Z := XY - YX$ is called the *bracket* of X and Y (see [4], pages 26-27), denoted by $[X, Y]$. Let

$$X(x) = \sum_i a_i(x) \frac{\partial}{\partial \xi_i}, \quad Y(x) = \sum_i b_i(x) \frac{\partial}{\partial \xi_i}.$$

Then from (3.1) we can derive that

$$[X,Y] = \sum_j \left(\sum_i a_i \frac{\partial b_j}{\partial \xi_i} - b_i \frac{\partial a_j}{\partial \xi_i} \right) \frac{\partial}{\partial \xi_j}.$$

Riemannian Manifold. A differentiable manifold with a given Riemannian metric is called a *Riemannian Manifold.* A Riemannian metric $\langle\ ,\ \rangle_x$ of M is a symmetric, bilinear and positive-definite form on the tangent space T_xM. Since M is a sub-manifold of Euclidean space $\mathbb{R}^k$, we use the *induced metric*:

$$\langle u, v\rangle_x = u^T v, \qquad u, v \in T_xM.$$

Connection. Let us indicate by $\mathcal{X}(M)$ the set all vector fields of class C^∞ on M and by $\mathcal{D}(M)$ the ring of real-valued functions of class C^∞ defined on M. An affine connection ∇ on a differentiable manifold M is a mapping $\nabla:\ \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ which is denoted by $(X,Y) \to \nabla_X Y$ and which satisfies the following properties:

1) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$.
2) $\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z$.
3) $\nabla_X (fY) = f\nabla_X Y + X(f)Y$,

in which $X, Y, Z \in \mathcal{X}(M)$ and $f, g \in \mathcal{D}(M)$.

Choose a system of coordinates (ξ_1, ξ_2),

$$X = \sum_i a_i t_i,\ Y = \sum_j b_j t_j,$$

where $t_i = \frac{\partial}{\partial \xi_i}$, then from properties 1)–3) we have

$$\nabla_X Y = \sum_k \left(\sum_{ij} a_i b_j \Gamma_{ij}^k + X(b_k) \right) t_k, \tag{3.2}$$

where Γ_{ij}^k is defined by

$$\nabla_{t_i} t_j = \sum_k \Gamma_{ij}^k t_k. \tag{3.3}$$

An affine connection ∇ on M is said to be symmetric if

$$\nabla_X Y - \nabla_Y X = [X,Y] \quad \text{for all} \quad X, Y \in \mathcal{X}(M).$$

A connection ∇ on a Riemannian manifold M is compatible with the metric $\langle\ ,\ \rangle$ if and only if ([4], page 54)

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \quad X, Y, Z \in \mathcal{X}(M).$$

3.2 Riemannian Curvature

Here we start with the Levi-Civita theorem ([4] page 55): *Given a Riemannian manifold M, there exists a unique affine connection ∇ on M satisfying the conditions:*

a) ∇ is symmetric.
b) ∇ is compatible with the Riemannian metric.

The connection defined by the Levi-Civita theorem is called the Riemannian connection. For the Riemannian connection, the number Γ_{ij}^k defined by (3.3), which is called the *Christoffel Symbols*, is calculated by

$$\Gamma_{ij}^m = \frac{1}{2}\sum_k \left\{ \frac{\partial g_{jk}}{\partial \xi_i} + \frac{\partial g_{ki}}{\partial \xi_j} - \frac{\partial g_{ij}}{\partial \xi_k} \right\} g^{km}, \tag{3.4}$$

where $(g^{kl}) = (g_{ij})^{-1}$, $(g_{ij}) = G$. Note that $\Gamma_{ij}^m = \Gamma_{ji}^m$, since $g_{ij} = g_{ji}$. It is easy to recognize that if M is an Euclidean space $\Gamma_{ij}^m = 0$.

Curvature. The *curvature* ([4], page 89) of a Riemannian manifold M is a correspondence which associates to every pair of vector fields X, Y a mapping $R(X,Y)$ which maps a vector field of M to another vector field of M given by

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, \tag{3.5}$$

where ∇ is the Riemannian connection of M.

Let (U_α, x_α) be a coordinate system at point $x \in M$. Let $\frac{\partial}{\partial \xi_i} = t_i$ and put

$$R(t_i, t_j)t_k = \sum_{l=1}^{2} R_{ijk}^l t_l. \tag{3.6}$$

Then $R(X,Y)Z$ can be expressed as

$$R(X,Y)Z = \sum_{i,j,k,l} R_{ijk}^l a_i b_j c_k t_l,$$

where $X = \sum_i a_i t_i$, $Y = \sum_j b_j t_j$, $Z = \sum_k c_k t_k$. From (3.2) we can derive that R_{ijk}^l are given as

$$R_{ijk}^s = \sum_l \Gamma_{ik}^l \Gamma_{jl}^s - \sum_l \Gamma_{jk}^l \Gamma_{il}^s + \frac{\partial \Gamma_{ik}^s}{\partial \xi_j} - \frac{\partial \Gamma_{jk}^s}{\partial \xi_i}. \tag{3.7}$$

Riemannian curvature. The counterpart of the Gaussian curvature for a surface is the *Riemannian curvature* for a Riemannian manifold ([4, 8]). For $x \in M$, let $X, Y \in T_x M$ be two linearly independent vectors. Then the *Riemannian curvature* of the tangent space $T_x M$ is defined by

$$K(x) = \frac{\langle R(X,Y)X, Y\rangle_x}{\|X\|^2 \|Y\|^2 - \langle X, Y\rangle_x^2}.$$

The Riemannian curvature, also called *sectional curvature*, is originally defined for a two-dimensional subspace of the tangent space $T_x M$. However, since $T_x M$ is assumed to be two-dimensional in this paper, the Riemannian curvature is then uniquely defined. For a regular surface in $\mathbb{R}^3$, the Riemannian curvature is the Gaussian curvature. It is not difficult to realize ([4], page 94) $K(x)$ does not depend on the choice of the vectors $X, Y \in T_x M$. Hence, we can use $X = t_1$, $Y = t_2$ and from (3.6)–(3.7) we have

$$\begin{aligned} \langle R(X,Y)X, Y\rangle_x &= \sum_{s=1}^{2} R_{121}^l g_{s2} \\ &= \sum_{s=1}^{2} \left[\sum_{l=1}^{2} \left(\Gamma_{11}^l \Gamma_{2l}^s - \Gamma_{21}^l \Gamma_{1l}^s \right) \right] g_{s2} + \sum_{s=1}^{2} \left(\frac{\partial \Gamma_{11}^s}{\partial \xi_2} - \frac{\partial \Gamma_{21}^s}{\partial \xi_1} \right) g_{s2}. \end{aligned} \tag{3.8}$$

It follows from (3.4) that

$$[\Gamma_{ij}^1,\ \Gamma_{ij}^2] = t_{ij}^T[t_1,\ t_2]G^{-1}, \quad i,j = 1,2. \tag{3.9}$$

Substituting these into (3.8) we have

$$\begin{aligned}\sum_{s=1}^{2}\left(\frac{\partial\Gamma_{11}^s}{\partial\xi_2} - \frac{\partial\Gamma_{21}^s}{\partial\xi_1}\right) g_{s2} \;&=\; (t_{11}^T t_{22} - t_{12}^T t_{12}) + t_{11}^T[t_1,t_2]G^{-1}\left\{[t_{12},t_{22}]^T t_2 + [t_1,t_2]^T t_{22}\right\} \\ &-\; t_{12}^T[t_1,t_2]G^{-1}[t_{11}^T t_2 + t_1^T t_{12}, t_{12}^T t_2 + -t_2^T t_{12}]^T.\end{aligned} \tag{3.10}$$

Using (3.9), the first summation of (3.8) can be written as

$$\begin{aligned}(\Gamma_{11}^2\Gamma_{22}^1 - \Gamma_{12}^1\Gamma_{12}^2)g_{12} \;&+\; [\Gamma_{12}^2(\Gamma_{11}^1 - \Gamma_{12}^2) + \Gamma_{11}^2(\Gamma_{22}^2 - \Gamma_{21}^1)]g_{22} \\ &=\; [\Gamma_{11}^1,\Gamma_{11}^2][t_{12}^T t_2, t_{22}^T t_2]^T - [\Gamma_{21}^1,\Gamma_{21}^2][t_{11}^T t_2, t_{12}^T t_2]^T.\end{aligned} \tag{3.11}$$

Combining (3.10) with (3.11) we arrive at formula (2.2). Note that (3.8) involves the third order partial derivatives of M, but (2.2) does not.

3.3 Mean Curvature

For a 2-dimensional Riemannian sub-manifold M of $\mathbb{R}^k$, the mean curvature vector is defined by ([9] page 119)

$$H(x) = \frac{1}{2}[h(e_1,e_1) + h(e_2,e_2)],$$

where (e_1,e_2) is an orthonormal frame for the tangent space to M at x. $h(X,Y)$ is defined by

$$h(X,Y) = \tilde{\nabla}_X Y - \nabla_X Y,$$

where ∇ and $\tilde{\nabla}$ are the Riemannian connection in M and $\mathbb{R}^k$, respectively. Since $\nabla_X Y \in TM$, $h(X,Y) \in TM^{\perp}$, we may consider only the computation of $\tilde{\nabla}_X Y$ and then project it into the normal space to obtain $h(X,Y)$. It follows from (3.2) that

$$\tilde{\nabla}_{e_l} e_l = \left[\frac{\partial e_l}{\partial x_1},\ \cdots,\ \frac{\partial e_l}{\partial x_k}\right] e_l, \tag{3.12}$$

where the fact $\Gamma_{ij}^l = 0$ for the Euclidean space $\mathbb{R}^k$ has been used. The orthonormal frame (e_1,e_2) can be obtained by the Gram-Schmitt process from (t_1,t_2):

$$e_1 = t_1/\sqrt{g_{11}}, \quad e_2 = (g_{11}t_2 - g_{12}t_1)/\sqrt{g_{11}\det(G)}, \tag{3.13}$$

Since $x_j = x_j(\xi_1,\xi_2),\ j = 1,\cdots,k$, we have

$$\varepsilon_j = t_1\frac{\partial\xi_1}{\partial x_j} + t_2\frac{\partial\xi_2}{\partial x_j}, \quad j = 1,\cdots,k, \tag{3.14}$$

where $\varepsilon_j = [0,\cdots,0,1,0,\cdots,0]^T$ is the j-th unit vector in $\mathbb{R}^k$. Performing the inner product of both sides of (3.14) with t_1 and t_2 and then solving the linear system derived for the unknowns $\frac{\partial\xi_1}{\partial x_j}, \frac{\partial\xi_2}{\partial x_j}$, we get

$$\left[\frac{\partial\xi_1}{\partial x_j}, \frac{\partial\xi_2}{\partial x_j}\right]^T = G^{-1}\left[t_1^T\varepsilon_j, t_2^T\varepsilon_j\right]^T.$$

Then by

$$\frac{\partial e_l}{\partial x_j} = \frac{\partial e_l}{\partial \xi_1}\frac{\partial \xi_1}{\partial x_j} + \frac{\partial e_l}{\partial \xi_2}\frac{\partial \xi_2}{\partial x_j}, \quad l = 1,2;\ j = 1,\cdots,k,$$

we have

$$\tilde{\nabla}_{e_l} e_l = \left[\frac{\partial e_l}{\partial \xi_1}, \frac{\partial e_l}{\partial \xi_2}\right] G^{-1}[t_1, t_2]^T e_l. \tag{3.15}$$

Taking $l = 1,2$ and using (3.13) we get

$$\tilde{\nabla}_{e_1} e_1 = \frac{\partial e_1}{\partial \xi_1}/\sqrt{g_{11}}, \quad \tilde{\nabla}_{e_2} e_2 = \left(-g_{12}\frac{\partial e_2}{\partial \xi_1} + g_{11}\frac{\partial e_2}{\partial \xi_2}\right)/\sqrt{g_{11}\mathrm{det}G}.$$

Since what we required is the part of $\tilde{\nabla}_{e_l} e_l$ orthogonal to the tangent space, we get

$$\left[\tilde{\nabla}_{e_1} e_1\right]^{\perp} = \frac{[t_{11}]^{\perp}}{g_{11}}, \quad \left[\tilde{\nabla}_{e_2} e_2\right]^{\perp} = \frac{[g_{12}^2 t_{11} + g_{11}^2 t_{22} - 2g_{11}g_{12}t_{12}]^{\perp}}{g_{11}\mathrm{det}G},$$

where $[\cdot]^{\perp}$ denotes the normal component of a vector. From this we have

$$H(x) = \frac{[g_{22}t_{11} + g_{11}t_{22} - 2g_{12}t_{12}]^{\perp}}{2(g_{11}g_{22} - g_{12}^2)}, \tag{3.16}$$

and thereafter (2.3) is derived, since Q is a projector that maps a vector to the normal space.

3.4 Principal Curvatures and Principal Directions

Since $M \subset \mathbb{R}^k$, the normal space, denoted by $T_xM^{\perp}$, can be defined at each point $x \in M$:

$$T_xM^{\perp} = \{n \in \mathbb{R}^k : \ \langle t, n\rangle_x = 0, \ \ \forall t \in T_xM\}.$$

Let n be a normal vector field on M and X be a vector field tangent to M. Then we have

$$\tilde{\nabla}_X n = -\mathcal{A}_n X + \nabla_X^{\perp} n,$$

where $-\mathcal{A}_n X$ and $\nabla_X^{\perp} n$ are respectively the tangent and the normal components. Then $\mathcal{A}_n$ is a self-adjoint map from TM to TM, called *second fundamental tensor* with respect to n ([9] pages 119-121). The *principal curvatures* $k_1(x)$, $k_2(x)$ and the *principal directions* $v_1(x)$, $v_2(x)$ with respect to n are defined as the eigenvalues and the orthonormal eigenvectors of $\mathcal{A}_n$. However, the principal curvatures and the principal directions are not uniquely defined since the normal vector field is not uniquely defined for $k > 3$. In this paper we have chosen a special normal vector field $h = H(x)/\|H(x)\|$, which is the normalized mean curvature vector field of the manifold M and is uniquely defined.

To calculate the spectrum of $\mathcal{A}_h$, we need to obtain its matrix representation. Let e_1, e_2 be the orthonormal basis of T_xM defined by (3.13):

$$[e_1, e_2] = [t_1, t_2]W, \ \ \text{with} \ \ W = \begin{bmatrix} g_{11}^{-\frac{1}{2}} & -g_{12}[g_{11}\det(G)]^{-\frac{1}{2}} \\ 0 & g_{11}[g_{11}\det(G)]^{-\frac{1}{2}} \end{bmatrix}.$$

Let

$$\mathcal{A}_h e_i = a_{1i}e_1 + a_{2i}e_2, \quad i = 1,2. \tag{3.17}$$

Then

$$\mathcal{A}_h[e_1, e_2] = [e_1, e_2]A_h,$$

where A_h is a 2×2 matrix which needs to be calculated in the following. It will be clear soon that A_h is symmetric. Before giving an explicit form of A_h, we first show that the eigenvalues of A_h are the eigenvalues of $\mathcal{A}_h$. Let

$$A_h = S \ \mathrm{diag}(\lambda_1, \lambda_2) \ S^T, \quad S^T S = I, \quad \text{and} \quad [v_1, v_2] = [e_1, e_2]S. \tag{3.18}$$

Then

$$\begin{aligned}\mathcal{A}_h[v_1, v_2] &= \mathcal{A}_h[e_1, e_2]S \\ &= [e_1, e_2]A_h S \\ &= [e_1, e_2]S\mathrm{diag}(\lambda_1, \lambda_2) \\ &= [v_1, v_2]\mathrm{diag}(\lambda_1, \lambda_2).\end{aligned}$$

Hence, v_i is the eigenvector of $\mathcal{A}_h$ with respect to the eigenvalue λ_i. Furthermore,

$$[v_1, v_2]^T[v_1, v_2] = S^T[e_1, e_2]^T[e_1, e_2]S = I.$$

That is, v_1, v_2 are orthonormal.

Now we calculate the matrix A_h. To this end, we need to calculate $\tilde{\nabla}_{e_i} h$. Paralleling to the derivation of $\tilde{\nabla}_{e_i} e_i$, we have an expression similar to (3.15):

$$\tilde{\nabla}_{e_l} h = \left[\frac{\partial h}{\partial \xi_1}, \frac{\partial h}{\partial \xi_2}\right] G^{-1}[t_1, t_2]^T e_l, \quad l = 1, 2. \tag{3.19}$$

If we project $\tilde{\nabla}_{e_l} h$ into the tangent space and express $\mathcal{A}_h e_l$ as (3.17), A_h can be expressed as

$$\begin{aligned}A_h &= -[e_1, e_2]\left[\frac{\partial h}{\partial \xi_1}, \frac{\partial h}{\partial \xi_2}\right] G^{-1}[t_1, t_2]^T[e_1, e_2] \\ &= -W^T[t_1, t_2]^T \left[\frac{\partial h}{\partial \xi_1}, \frac{\partial h}{\partial \xi_2}\right] W.\end{aligned} \tag{3.20}$$

Now we need to calculate $[t_1, t_2]^T \left[\frac{\partial h}{\partial \xi_1}, \frac{\partial h}{\partial \xi_2}\right]$. Substituting the expression h into (3.20) and with some additional calculations, we have

$$A_h = W^T F_h W, \tag{3.21}$$

where F_h is a 2×2 symmetric matrix defined by

$$F_h = \frac{g_{22}(B_{11} - A_{11}) + g_{11}(B_{22} - A_{22}) - 2g_{12}(B_{12} - A_{12})}{2\det(G)\|H(x)\|}, \tag{3.22}$$

$$A_{kl} = (t_{ij}^T t_{kl})_{ij=1}^2, \quad B_{kl} = (c_{ij}^T G^{-1} c_{kl})_{ij=1}^2, \quad c_{ij} = [t_1, t_2]^T t_{ij}. \tag{3.23}$$

Substituting (3.23) into (3.22) and using the mean curvature formula (2.3), we have a simple expression for F_h:

$$F_h = -\left(t_{ij}^T \, h(x)\right)_{ij=1}^2.$$

Having an explicit expression for A_h, we are able to compute the principal curvatures and the principal directions by (2.4), (2.5) and (3.18). However, since A_h involves W, it is not intrinsic. To obtain more elegant formulas, we rewrite $A_h = W^T F_h W$ as follows

$$A_h = (\Lambda^{\frac{1}{2}} K W)^T A (\Lambda^{\frac{1}{2}} K W) \ \text{ with } \ A = \Lambda^{-\frac{1}{2}} K F_h K^T \Lambda^{-\frac{1}{2}}.$$

It follows from (2.6) and (2.7) that $[u_1, u_2]^T[u_1, u_2] = I$. We then have

$$\begin{aligned} I &= [e_1, e_2]^T[e_1, e_2] \\ &= W^T[t_1, t_2]^T[t_1, t_2]W \\ &= W^TK^T\Lambda^{\frac{1}{2}}[u_1, u_2]^T[u_1, u_2]\Lambda^{\frac{1}{2}}KW \\ &= (\Lambda^{\frac{1}{2}}KW)^T(\Lambda^{\frac{1}{2}}KW), \end{aligned} \tag{3.24}$$

that is, $\Lambda^{\frac{1}{2}}KW$ is an orthgonal matrix. Hence, the eigenvalues of A_h and A are the same, and therefore we can use A to compute the principal curvatures instead of A_h. Using relation (2.8), we have

$$A_h = (\Lambda^{\frac{1}{2}}KW)^T P\mathrm{diag}(k_1, k_2)P^T(\Lambda^{\frac{1}{2}}KW).$$

Hence S could be written as

$$S = (\Lambda^{\frac{1}{2}}KW)^T P.$$

Therefore, the eigenvectors are given by

$$\begin{aligned} [v_1, v_2] &= [e_1, e_2]S \\ &= [t_1, t_2]WS \\ &= [u_1, u_2](\Lambda^{\frac{1}{2}}KW)(\Lambda^{\frac{1}{2}}KW)^T P \\ &= [u_1, u_2]P, \end{aligned}$$

and hence (2.9) is derived.

Remark 1. Since h uses the second order partial derivatives of $x \in M$, $\frac{\partial h}{\partial \xi_i}$ uses the third order partials. A nice property is that all the third order partials are canceled in A_h. The final result only uses the first and the second order partials.

Remark 2. Both the matrix A for computing the principal curvatures and the formula (2.9) for computing the principal directions do not involve W. Furthermore, it can be proved that all the curvatures do not depend on the choice of the local coordinate system (ξ_1, ξ_2). Therefore, they are intrinsic to the manifold M. The proof of this claim is not the theme of this paper.

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