PHL 313K

Goals

The top-level goal of this class is to give students the ability to use the formal language of mathematics to help them solve problems. The students should develop two key abilities:

1. To map back and forth between formal statements on paper and ideas in their head, and
2. To use the definitions of functions, relations, and sets, the standard axioms of arithmetic and set theory, and the rules of inference of both propositional and predicate logic to enable them to derive new theorems, and hopefully thus insight, from an initial statement of a problem.

Syllabus

1. Sentential Calculus (SC) [3 weeks]:
    - 1.1. The basics: syntax, semantics, satisfiability, tautological consequence, tautologies, SC formal proofs.
    - 1.2. Translating ideas to logic and back again
2. First Order Predicate Calculus (PC) [3 weeks]:
    - 2.1. Syntax, semantics,
    - 2.2. Satisfiability, consequence, valid sentences, simple PC formal proofs.
    - 2.3. Genuine understanding of quantifiers, particularly when they are nested
    - 2.4. Translating ideas to logic and back again using quantifiers.
3. Sets [2 weeks]:
    - 3.1. Definition of set and of the basic set operations
    - 3.2. Translating set definitions to formal statements and back again
    - 3.3. Use of Venn diagrams to visualize set operations
    - 3.4. Cardinality (of finite sets). Optional: cardinality of infinite sets.
    - 3.5. Theorems about sets and their operations (associativity, commutativity, etc.)
    - 3.6. Cartesian products of sets
    - 3.7. Proving theorems about sets
        - 3.7.1. Syntactic proofs
        - 3.7.2. Semantic proofs (reasoning about the elements of the sets)
4. Relations [2 weeks]
    - 4.1. Definition of a relation
    - 4.2. Binary relations
        - 4.2.1. Properties: reflexive, symmetric, transitive, antisymmetric
        - 4.2.2. Partial and total orderings of sets defined by a relation
        - 4.2.3. Equivalence relations and partitions
5. Functions [2 weeks]:
    - 5.1. Definition
    - 5.2. Properties of functions (one-to-on, onto, bijection)
    - 5.3. Optional: pigeonhole principle
    - 5.4. Inverses
    - 5.5. Identities
    - 5.6. Composition
6. Recursive Definitions and Mathematical Induction [1 week]:
    - 6.1. The natural number system; definition by recursion; definition of plus and times for natural numbers; proofs of basic properties of plus and times; ordering the natural numbers.
    - 6.2. Proof by induction
        - 6.2.1. For theorems about the natural numbers

6.2.2. For other things, e.g., sets

7. Additional elementary topics involving logic, set theory, or induction. For example, conjunctive normal forms and SC resolution, PC with identity and terms, ordering relations, additional elementary number theory, well-orderings, course of values induction, induction on strings, lists and trees.

Problems that Students Should be Able to Solve

All students should be able to solve all the starred problems. Students should be able to get at least 80% of the remaining problems.

Some problems will ask the student to use the # vocabulary. It is given at the end of this problem set.

1. Sentential Calculus (SC)

1) * Prove that for all propositions p, q, and r:
   a) $[\sim q \Rightarrow \sim p] \Rightarrow [p \Rightarrow q]$
   b) $[p \Rightarrow (q \wedge r)] \Rightarrow [\sim p \vee (q \vee r)]$

2. First Order Predicate Calculus (PC)

This is a very difficult area for students. We find that they have a very weak intuitive grasp of quantifiers, particularly when they are nested.

1) * State an assertion that, if true, would falsify each of the following claims:
   a) All zamzows have tenockritus.
   b) Some zamzows have tenockritus.
2) Given the following two axioms:
   $\forall x\ Px \Rightarrow Qx$
   $\exists x \sim Qx \wedge \sim Rz$
   Prove that
   $\exists y \sim Py$
3) Consider the predicates *Wx* (*x* is a woman), *CHILDOFxy* (*y* is a child of *x*), and *Mx* (*x* is a mother).
   a) * Write a PC formula to describe the fact that all women with children are mothers.
   b) Modify the definition of mother given in (a) to add the fact that *only* women with children are mothers. Use this, plus the fact that Mary has no children, to prove that Mary is not a mother.
   c) Specify a predicate *P* that makes the following true given the universe of people. You must state a nontrivial predicate (i.e., True is not an acceptable answer):
      $\forall x \sim Px \Rightarrow \exists y,z\ CHILDOFyx \wedge CHILDOFyz \wedge x \neq z$
4) Write the negation of
      $\exists m\ \exists n\ \forall x\ Pxn \Rightarrow Qxm$
5) * Write PC statements to express each of the following facts. You may use any of the following predicates:
   - $Exy \equiv x$ has sent an email message to y.
   - $Txy \equiv x$ has telephoned y.
   - $Cxyr \equiv x$ has chatted with y in on-line chat room r.
   - $Sxy \equiv x$ has taken course y.
   - $Oxy \equiv$ department x offers course y.

   Make sure that you state the universe of discourse for each quantified variable you use.

a) There are two students in your school who, between them, have emailed or telephoned everyone else in the school.
b) Every student in your school has chatted with at least one other student in at least one on-line chat room.
c) There is a student in your school who has not received an email message from anyone else in the school.
d) There is a student in your school who has taken every course offered by one of the departments in this school.
e) There is a department in your school from which no student has ever taken a course.

6) Let the universe of discourse for x and y be the positive integers. Define $GTEyx \equiv (y \geq x)$. Give a counterexample to the following assertion:

   $\forall x\ ((\exists y\ GTEyx) \Rightarrow (\exists y \sim GTEyx))$

7) * Let the universe of discourse for x and y be the set of people. Let Fxy be true iff x and y are friends. Consider the following statement S:

   $\forall x\ \exists y\ Fxy$

   a) Translate S into English.
   b) Give a PC formula for ~S.
   c) Translate your answer to b) into English.

8) Let the universe of discourse for x, y, and z be the set of students at your school. Let Fxy be true iff x and y are friends. Translate the following statement into English:

   $\exists x\ \forall y\ \forall z\ (Fxy \wedge Fxz \wedge y \neq z) \Rightarrow \sim Fyz$

3. Sets

1) * What are these sets? Write them using braces, commas, numerals, … (for infinite sets), and $\varnothing$ only. N is the set of natural numbers. ($\mathcal{P}(S)$ is the power set of S.)
   a) $(\{1, 3, 5\} \cup \{3, 1\}) \cap \{3, 5, 7\}$
   b) $\cup\{\{A\}, \{A, 5\}, \cap\{\{5, 7\}, \{7, 9\}\}\}$
   c) $(\{1, 2, 5\} - \{5, 7, 9\}) \cup (\{5, 7, 9\} - \{1, 2, 5\})$
   d) $\{1\} \cup \{\varnothing\} \cup \varnothing$
   e) $\mathcal{P}(\{7, 8, 9\}) - \mathcal{P}(\{7, 9\})$
   f) $\mathcal{P}(\varnothing)$
   g) $\{x : x \text{ is an integer and } x^2 = 2\}$
   h) $\{x : \exists y \in N \text{ where } x = y^2\}$
   i) $\{1\} \times \{1, 2\} \times \{1, 2, 3\}$
   j) $\varnothing \times \{1, 2\}$
   k) $\mathcal{P}(\{1, 2\}) \times \{1, 2\}$

2) * What is the cardinality of each of the following sets? Justify your answer. N is the set of natural numbers (*i.e.*, the non-negative integers.)
   a) $S = \{\varnothing, \{\varnothing\}\}$
   b) $S = \mathcal{P}(\{a, b, c\})$
   c) $S = \{a, b, c\} \times \{1, 2, 3, 4\}$

3) * Consider the following set manipulation problems:
   a) Let $S = \{a, b\}$. Let $T = \{b, c\}$. List the elements of P, defined as

      $P = \mathcal{P}(S) \cap \mathcal{P}(T)$.

   b) Let N be the set of nonnegative integers.

      Let $S = \{x \in Z: \exists y \in N \text{ where } x = 2y\}$.

      Let $T = \{x \in Z: \exists y \in N \text{ where } x = 2^y\}$.

Let $W = S - T$. Describe W in English. List any five consecutive elements of W.
Let $X = T - S$. What is X?

4) * Prove the following for all sets, A, B, C, and D. Do this either syntactically, using the set identities, or semantically, by writing logical assertions that must be true of the elements in the two sets.
   a) $(A \cap B) - C \subseteq (A \cup D) \cap B$.
   b) $(A - B) - C = A - (B \cup C)$
5) * Clearly describe the difference between $\varnothing$ and $\{\varnothing\}$. What is the cardinality of each of these sets?
6) * Let $S = \{a, b, c, d\}$. Let $X = \{A \subseteq S : a \in A \rightarrow b \notin A\}$. List all elements of X.
7) * Let $S = \{a, b, c, d\}$. Let $Y = \{s \in S : s \neq b\}$. List all elements of Y.
8) * Let $S = \{a, b, c, d\}$. Let $T = \{b, c, x, y, z\}$. Let $Z = \{w \subseteq T : w \subseteq S\}$. List all elements of Z.
9) * Let $S = \{a, b, c, d\}$. Let $T = \{b, c, x, y, z\}$. Let $Z' = \{w \subseteq T : w \cap S = \varnothing\}$. List all elements of Z'.
10) * Suppose in a class, 26 students got an A on Exam 1 and 21 got an A on Exam 2. If 30 got an A on at least one of the two exams, how many got A's on both exams?
11) Let Z denote the set of integers. Using the # vocabulary, describe the set of even integers greater than 5.
12) Given a set X of subsets of a set S, define, using the # vocabulary, the set of elements of X that have cardinality equal to 2 (i.e., contain exactly two objects from S).
13) Given a set X of subsets of a set S, define, using the # vocabulary, the set of elements from S that appear in exactly one element of X.
14) Let A, B be two sets. If $\mathcal{P}(A) = \mathcal{P}(B)$ must $A = B$? Prove your answer.

4. Relations

1) * For each of the following sets, state whether or not it is a partition of {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.
   a) {{0}, {1}, {2}, {3}, {4}, {5}, {6}, {7}, {8}, {9}, {10}}
   b) {∅, {1}, {2}, {3}, {4}, {5}, {6}, {7}, {8}, {9}, {10}}
   c) {{1, 2}, {3, 4}, {5, 6}, {7, 8}, {9, 10}}
   d) {{1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 6}, {6, 7}, {7, 8}, {8, 9}, {9, 10}}
2) * For each of the following relations, state which of these properties hold:reflexivity, symmetry, transitivity, and antisymmetry.
   a) = defined on strings
   b) ≠ defined on strings
   c) < defined on N (the natural numbers)
   d) subset of defined on the power set of N
3) * For each of the following relations R, over some domain D, compute the reflexive, symmetric, transitive closure R'. Try to think of a simple descriptive name for the new relation R'. Since R' must be an equivalence relation, describe the partition that R induces on D.
   a) Let D be the set of 50 states in the US. $\forall xy$, xRy iff x shares a boundary with y.
   b) Let D be the natural numbers. $\forall xy$, xRy iff $y = x+3$.
   c) Let D be the set of strings containing no symbol except a. $\forall xy$, xRy iff $y = xa$. (i.e., if y equals x concatenated with a).
4) Consider an infinite rectangular grid (like an infinite sheet of graph paper). Let S be the set of intersection points on the grid. Let each point in S be represented as a pair of (x,y) coordinates where adjacent points differ in one coordinate by exactly 1 and coordinates

increase (as is standard) as you move up and to the right. (In other words, the standard (x,y) grid with integer coordinates.)

a) Let R be the following relation on S: $\forall(x_1,y_1)(x_2,y_2)$, $(x_1,y_1)R(x_2,y_2)$ iff $x_2 = x_1+1$ and $y_2=y_1+1$. Let R' be the reflexive, symmetric, transitive closure of R. Describe in English the partition P that R' induces on S. What is the cardinality of P?

b) Let R be the following relation on S: $\forall(x_1,y_1)(x_2,y_2)$, $(x_1,y_1)R(x_2,y_2)$ iff $(x_2 = x_1+1$ and $y_2=y_1+1)$ or $(x_2 = x_1-1$ and $y_2=y_1+1)$. Let R' be the reflexive, symmetric, transitive closure of R. Describe in English the partition P that R' induces on S. What is the cardinality of P?

c) Let R be the following relation on S: $\forall(x_1,y_1)(x_2,y_2)$, $(x_1,y_1)R(x_2,y_2)$ iff (x_2,y_2) is reachable from (x_1,y_1) by moving two squares in any one of the four directions and then one square in a perpendicular direction. Let R' be the reflexive, symmetric, transitive closure of R. Describe in English the partition P that R' induces on S. What is the cardinality of P?

5) Let A be a set of people. Let F be the friendship relation on A. In other words, xFy iff x is friends with y. We will say that A is a "friendly bunch of people" if everyone in A is friends with at least as many people as they are not friends with. Using the # vocabulary, define this predicate formally.

6) Write a logical expression that describes the set of elements x of a set S partially ordered by $\geq$, such that there are exactly two elements in S, other than x, that are greater than or equal to x.

7) For each of the following relations, state whether it is a partial order (that is not also total), a total order, or neither. Justify your answer.

a) *DivisibleBy*, defined on the natural numbers. $(x, y) \in DivisibleBy$ iff x is evenly divisible by y. So, for example, $(9, 3) \in DivisibleBy$ but $(9, 4) \notin DivisibleBy$.

b) *LessThanOrEqual* defined on ordered pairs of natural numbers. $(a, b) \leq (x, y)$ iff $a \leq x$ or $(a = x$ *and* $b \leq y)$. For example, $(1,2) \leq (2,1)$ and $(1,2) \leq (1,3)$.

5. Functions

1) * Let $R = \{(a, b), (a, c), (c, d), (a, a), (b, a)\}$. What is $R \circ R$, the composition of R with itself? What is R^{-1}, the inverse of R? Is R, $R \circ R$, or R^{-1} a function?

2) * For each of the following functions, state whether or not it is (i) one-to-one, and (ii) onto. Justify your answers. Let Z be the set of integers, N be the set of nonnegative integers, and P be the set of positive integers.

a) $F: Z \rightarrow N$, where
$F(x) = 1 + x^2$

b) $G: N \rightarrow N$, where
$G(x) = x + 1$

c) $H: Z \rightarrow N$, where
$H(x) = x^2$

d) $K: Z \rightarrow Z$
$K(x) = -x$

e) $+: P \times P \rightarrow P$, where
$+(a, b) = a + b$ (In other words, simply addition defined on the positive integers)

f) $X : B \times B \rightarrow B$, where B is the set {True, False} and
$X(a, b)$ = the exclusive or of a and b

3) * Let D be the set of people. For each of the following relations answer these questions: (i) Is it reflexive? (ii) Is it symmetric? (iii) Is it transitive? (iv) Is it a function? (v) If it is a function, is it one-to-one? (vi) If it is a function, is it onto?

a) $\forall x,y \in D\ xCy$ iff y is a child of x
b) $\forall x,y \in D\ xMy$ iff y is the mother of x
c) $\forall x,y \in D\ xPy$ iff y is a parent of x
d) $\forall x,y \in D\ xRy$ iff y is a blood relative of x

4) Define $(f \circ g)(x) \equiv f(g(x))$. Define the following functions on the integers:
   $d(x) = 2x$ (double)
   $v(x) = -x$ (invert)
   $a(x, y) = x + y$ (addition)
   $f(x) = a((d \circ v)(x), 10)$
   a) What is $f(3)$?
   b) Is f one-to-one? Explain.
   c) Is f onto? Explain.
   d) Is the inverse of f a function? If it is, what is its domain? What is its range? Explain.
5) Using the # vocabulary, write a logical formula to express the property that a function $f: A \rightarrow B$ is $1 - 1$.
6) Using the # vocabulary, write a logical formula to express the property that a function $f: A \rightarrow B$ is onto.
7) Are the following sets closed under the following operations? If not, what are the respective closures?
   a) The odd integers under multiplication.
   b) The positive integers under division.
   c) The negative integers under subtraction.
   d) The negative integers under multiplication.
   e) The odd length strings under concatenation.

6. Recursive Definitions and Mathematical Induction

1. * Prove that for all integer values of $n \geq 0$, $\sum_{i=0}^{n} i = n(n+1)/2$
2. Prove that for all nonnegative integers n, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. (Recall that an empty summation has the value zero.)
3. For n ≥ 2, let A1, A2, ..., An be a collection of n sets. Using induction, prove the n-fold generalization of the DeMorgan Law:

$$\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i}$$

(You may assume that $\bigcup_{i=1}^{n+1} B_i = (\bigcup_{i=1}^{n} B_i) \cup B_{n+1}$ and $\bigcap_{i=1}^{n+1} B_i = (\bigcap_{i=1}^{n} B_i) \cap B_{n+1}$ for any collection of n+1 sets.)

Allowable Notation for Problems with (#)

You may use any of the following symbols in your answer:

- Numbers
- Letters, to represent objects (*e.g.*, sets or elements of sets)
- Standard logical symbols, including $\wedge$, $\vee$, $\sim$, $\Rightarrow$, $\equiv$.
- Standard notation for sets, including {, }, : (such that), $\in$, $\notin$, $\subseteq$, $\cup$, $\cap$, $\varnothing$, and $\overline{}$ (complement)
- $|A|$ for the cardinality of A
- $A \times B$ for the cross product of A and B (and its extension to a k-way cross product)
- (and), both as delimiters and to indicate an ordered tuple (*e.g.*, *(a, b, c)*)
- 2^S or $\mathcal{P}(S)$ to indicate the power set of S
- Comparison operators, including $\leq$, $\geq$, $=$, $\neq$
- Arithmetic operators, including +, -, *, /
- Quantifiers: $\forall$ and $\exists$
- Predicates expressed as Px for P holds of x