CS303E: Elements of Computers and Programming

Recursion

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Last updated: August 29, 2024 at 13:58

Factorial Function

Consider the factorial function:

$$k! = 1 * 2 * \ldots * k.$$

This is often defined mathematically by the following *recurrence relation*:

$$\begin{aligned} 1! &= 1 \\ n! &= n * (n-1)!, \text{ for } n > 1 \end{aligned}$$

What assumptions are being made about the input?

We say that this definition is *recursive* because in defining the factorial function *we're using the factorial function*.

It is typically quite easy to implement a function in Python directly from the recurrence relation.

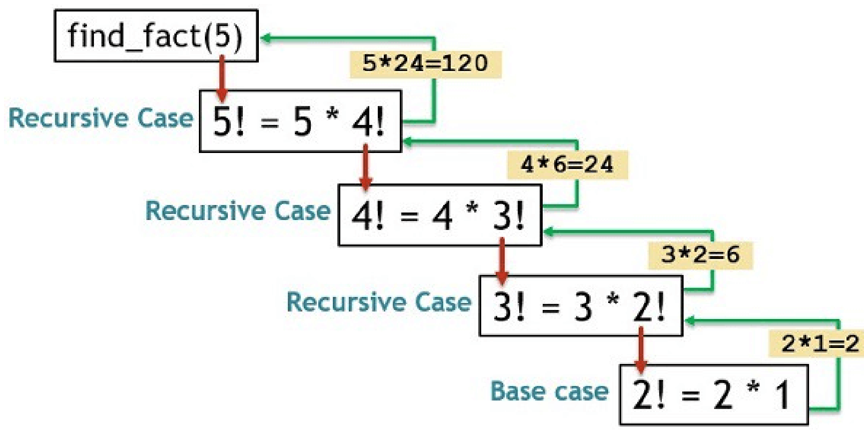
Factorial Function

$$\begin{aligned} 1! &= 1 \\ n! &= n * (n-1)!, \text{ for } n > 1 \end{aligned}$$

For example, to compute 5! we do the following:

$$\begin{aligned} 5! &= 5 * 4! & (1) \\ &= 5 * (4 * 3!) & (2) \\ &= 5 * (4 * (3 * 2!)) & (3) \\ &= 5 * (4 * (3 * (2 * 1!))) & (4) \\ &= 5 * (4 * (3 * (2 * 1))) & (5) \\ &= 5 * (4 * (3 * 2)) & (6) \\ &= 5 * (4 * 6) & (7) \\ &= 5 * 24 & (8) \\ &= 120 & (9) \end{aligned}$$

Factorial Function



Factorial Function

Mathematical definition:

$$1! = 1$$
$$n! = n * (n-1)!, \text{ for } n > 1$$

Here's a straightforward implementation in Python.

```
def fact (n):
    """ Factorial function. """
    if n == 1:
        return 1
    else:
        return n * fact(n-1)     # note recursive call
```

This function is **recursive** because it calls itself.

Can you see anything wrong with this? How might you fix it?

Python Factorial Function

How should you deal with an illegal input, say `fact(0)`?

Python Factorial Function

How should you deal with an illegal input, say `fact(0)`?

You can do everything you do for non-recursive functions:

1. Assume the input will be legal (and crash if it's not).
2. Print an error message and return.
3. Use Python error handling; we haven't covered that this semester!
4. Return a reasonable default answer.

```
def fact (n):
    """ Factorial function. """
    if n <= 1:
        return 1
    else:
        return n * fact(n-1)     # note recursive call
```


Recursion

In programming, recursion just means that a program calls itself, either directly or *indirectly*.

Functions A and B are *mutually recursive* if A calls B and B calls A.

```
def isNonnegativeEven ( n ):
    print("In isNonnegativeEven(", n, ")" )
    if ( n < 0 ):
        return False
    elif ( n == 0 ):
        return True
    else:
        return isNonnegativeOdd ( n - 1 )

def isNonnegativeOdd( n ):
    print("In isNonnegativeOdd(", n, ")" )
    if ( n < 1 ):
        return False
    elif ( n == 1 ):
        return True
    else:
        return isNonnegativeEven( n - 1 )
```

Mutual Recursion

```
>>> isNonnegativeOdd( 13 )
In isNonnegativeOdd( 13 )
In isNonnegativeEven( 12 )
In isNonnegativeOdd( 11 )
In isNonnegativeEven( 10 )
In isNonnegativeOdd( 9 )
In isNonnegativeEven( 8 )
In isNonnegativeOdd( 7 )
In isNonnegativeEven( 6 )
In isNonnegativeOdd( 5 )
In isNonnegativeEven( 4 )
In isNonnegativeOdd( 3 )
In isNonnegativeEven( 2 )
In isNonnegativeOdd( 1 )
True
>>>
```

There could be a cycle of three or more functions: A calls B, B calls C, C calls A. And so on!

Recursive Thinking

Recursion is also a *way of thinking about computing problems*: Solve a "big" problem by solving "smaller" instances of the *same* problem. The simplest instances can be solved directly.

Example: Suppose I decide to walk to the store.

Base case: I'm already there; there's nothing left to do.

Recursive case: I take one step; now I've reduced the problem to the "smaller" but structurally identical problem of walking from where I am now to the store.

Recursive Thinking

What can go wrong:

1. I don't know how to take a step.
2. I walk in the wrong direction.
3. There is no store.
4. You walk to the wrong store.
5. I can't recognize when I get to the store.
6. I walk right past the store and keep going forever.

Recursion

Any recursive function must have:

- one or more *base cases* that return an answer without calling the procedure recursively;
- one or more *recursive cases* that call the function on arguments that *move the computation in the direction of some base case*;
- assurance that you will eventually hit one of the base cases.

```
def fact (n):
    if n <= 1:                # the base case
        return 1
    else:
        return n * fact(n-1)  # the recursive case
```

How do you know that this eventually terminates?

Some Faulty Examples

```
def factBad( n ):
    # There is no store!
    return n * factBad( n - 1 )

def isEven( n ):
    # I may walk right past the store and
       keep going!
    if n == 0:
        return True
    else:
        return isEven(n - 2)
```

What's wrong and how would you fix these?

Recursive Thinking: Counting in a List

Example: Suppose you want to count the number of items in a list.

What's the base case? What's the simplest list you can think of?

Recursive Thinking: Counting in a List

Example: Suppose you want to count the number of items in a list.

What's the base case? What's the simplest list you can think of?
An empty list! How many items are in an empty list?

BTW: what's wrong with saying "a list with one element"?

Recursive Thinking: Counting in a List

If we're not in the base case, then what do we know?

Recursive Thinking: Counting in a List

If we're not in the base case, then what do we know?

We know that our input list has at least one element in it! *But we still don't know how many.*

But we could figure that out if we only had the solution to a slightly simpler problem: Suppose we knew how many items were in a list that was one shorter.

Then we'd be done. Why?

BTW, what is a list that is just our original list L with one element missing? That's just `L[1:]`.

Recursive Thinking: Counting in a List

Problem: how many items are in a list L?

Base case: If L is empty, length is 0.

Recursive case: Assume I know the length of L[1:], and use that to compute the length of L. How?

```
def countItemsInList( L ):
    """ Recursively count the number of items in list. """
    if not L:          # empty list counts as False
        return 0
    else:
        return 1 + countItemsInList( L[1:] )
```

```
>>> l1 = [ 1, 2, 3, "red", 2.9 ]
>>> countItemsInList( l1 )
5
```

Does this work for any list?

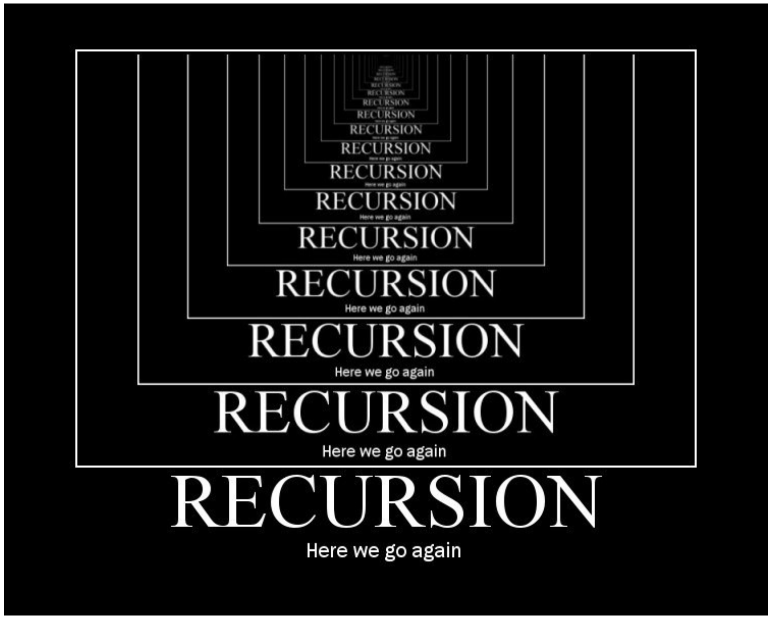
What's Actually Happening?

I instrumented the code to print out an "execution trace."

```
def countItems2 (lst, k):
   if not lst:
      print(" 1 +"*k, "+ 0 =")
      return 0
   else:
      print(" 1 +"*k, "1 + countItems(", lst[1:], ") =")
      return 1 + countItems2( lst[1:], k+1 )
```

```
>>> lst = [4, 5, 2, 5, 9, 2, 8]
>>> countItems2( lst, 0 )
 1 + countItems( [5, 2, 5, 9, 2, 8] ) =
 1 + 1 + countItems( [2, 5, 9, 2, 8] ) =
 1 + 1 + 1 + countItems( [5, 9, 2, 8] ) =
 1 + 1 + 1 + 1 + countItems( [9, 2, 8] ) =
 1 + 1 + 1 + 1 + 1 + countItems( [2, 8] ) =
 1 + 1 + 1 + 1 + 1 + 1 + countItems( [8] ) =
 1 + 1 + 1 + 1 + 1 + 1 + 1 + countItems( [] ) =
 1 + 1 + 1 + 1 + 1 + 1 + 1 + 0 =
7
```


It Seems Like Magic, but It's Not



Let’s Take a Break

Recursive Thinking: Some Examples

Example: how can you sum a list of numbers?

Base case: If L is empty, the sum is 0.

Recursive case: If I knew the sum of L[1:], then I could compute the sum of L. How?

```
def sumItemsInList( L ):
    """ Recursively sum the items in a list. """
    if not L:          # empty list counts as False
        return 0
    else:
        return L[0] + sumItemsInList( L[1:] )
```

```
>>> lst = [ 5, 6, 14, -3, 0, -70 ]
>>> sumItemsInList( lst )
-48
```


Recursive Thinking: Some Examples

Example: how can you count the occurrences of key in list L?

Base case: If L is empty, the count is 0.

Recursive case: If L starts with key, then it’s 1 plus the count in the rest of the list; otherwise, it’s just the count in the rest of the list.

```
def countOccurrencesInList( key, L ):
    """ Recursively count the occurrences of key in L. """
    if not L:          # empty list counts as False
        return 0
    elif key == L[0]:
        return 1 + countOccurrencesInList( key, L[1:] )
    else:
        return countOccurrencesInList( key, L[1:] )
```

```
>>> lst = [ 5, 6, 14, -3, 0, -70, 6 ]
>>> countOccurrencesInList( 3, lst )
0
>>> countOccurrencesInList( 6, lst )
2
```


Recursive Thinking: Some Examples

Example: how can you reverse a list L?

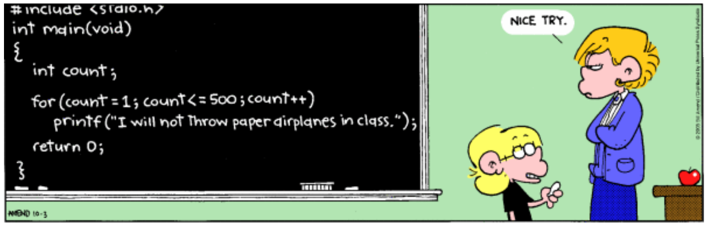
Base case: If L is empty, the reverse is [].

Recursive case: If L is not empty, remove the first element and append it to end of the reverse of the rest.

```
def reverseList( L ):
    """ Recursively reverse a list. """
    if not L:          # empty list counts as False
        return []
    else:
        return reverseList( L[1:] ) + [ L[0] ]
```

```
>>> lst = [ 1, 5, "red", 2.3, 17 ]
>>> print( reverseList( lst ))
[17, 2.3, 'red', 5, 1]
```

Recursive Thinking: Some Examples



How would Jason do this recursively?

```
COUNT = 500
STRING = "I will not throw paper airplanes in class."

def blackboard( n ):
    if n <= 0:                    # base case
        return
    else:
       print( STRING )            # recursive case
       blackboard( n - 1 )

blackboard( COUNT)
```


Recursive Thinking: Some Examples

An algorithm that dates from Euclid finds the greatest common divisor of two positive integers:

$$gcd(a, b) = a, \text{ if } a = b$$
$$gcd(a, b) = gcd(a, b - a), \text{ if } a < b$$
$$gcd(a, b) = gcd(a - b, b), \text{ if } b < a$$

```
def gcd( a, b ):
    """ Euclid's algorithm for GCD. """
    print( "Computing gcd(", a, ",", b, ")" )
    if a < b:
        return gcd( a, b-a )
    elif b < a:
        return gcd( a-b, b )
    else:
        print( "Found gcd:", a )
        return a

print("gcd( 68, 119 ) =", gcd( 68, 119 ))
```

What is assumed about a and b? What is the base case? The recursive cases?

Running GCD

```
> python gcd.py
Computing gcd( 68 , 119 )
Computing gcd( 68 , 51 )
Computing gcd( 17 , 51 )
Computing gcd( 17 , 34 )
Computing gcd( 17 , 17 )
gcd( 68, 119 ) = 17
```


Some Exercises for You to Try

1. Write a recursive function to append two lists.
2. Write a recursive version of linear search in a list.
3. Write a recursive version of binary search in a list.
4. Write a recursive function to sum the digits in a decimal number.
5. Write a recursive function to check whether a string is a palindrome.

It's probably occurred to you that many of these problems were already solved with built in Python methods or could be solved with loops.

That's true, but our goal is to teach you to think recursively!

Helper Functions

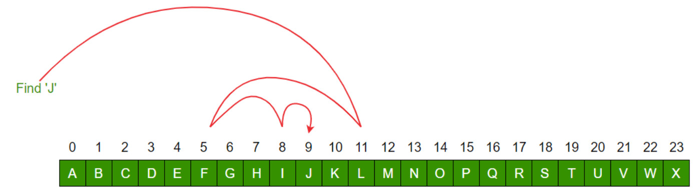
For some recursive solutions you'll need a **helper** function.

For example, remember **binary search** from slideset 10.

```
def BinarySearch( lst, key ):
    """ Search for key in sorted list lst. """
    low = 0
    high = len(lst) - 1
    while (high >= low):
        mid = (low + high) // 2
        if key < lst[mid]:
            high = mid - 1
        elif key == lst[mid]:
            return mid
        else:
            low = mid + 1
    # What's true here?  Why this value?
    return (-low - 1)
```

It's clear how to think of this recursively.

Binary Search



Here's one version:

```
def BinarySearchRecursive( lst, key ):
    """ Search for key in sorted list lst. """
    if lst == []:
        return False
    mid = ( len(lst) - 1 ) // 2
    if key == lst[mid]:
        return True
    elif key < lst[mid]:
        return BinarySearchRecursive( lst[:mid], key )
    else:
        return BinarySearchRecursive( lst[mid+1:], key )
```


Recursive Binary Search

```
>>> from BinarySearch import *
>>> lst = [ 2, 4, 7, 9, 10, 12, 14, 17, 20 ]
>>> BinarySearchRecursive( lst, 10 )
True
>>> BinarySearchRecursive( lst, 11 )
False
>>> BinarySearchRecursive( lst, 21 )
False
```

This is inferior to the iterative version for at least two reasons:

1. it creates a new copy of half of the list in each recursive call (by slicing);
2. it returns a Boolean rather than an index. Can you see why returning an index would be difficult?

A better approach would be to preserve the original list and add some parameters to the recursive function. This is often done with a **helper** function.

Recursive Binary Search with Helper

```
def BinarySearchHelper( lst, key, low, high ):
    if low > high:
        return -low - 1

    mid = (low + high) // 2
    if key == lst[mid]:
        return mid
    elif key < lst[mid]:
        return BinarySearchHelper(lst, key, low, mid - 1)
    else:
        return BinarySearchHelper(lst, key, mid + 1, high)

def BinarySearchRecursive2( lst, key ):
    """ Search for key in sorted list lst. """
    low = 0
    high = len(lst) - 1
    return BinarySearchHelper( lst, key, low, high )
```

Compare the helper to the nonrecursive version.

Running It

```
>>> from BinarySearch import *
>>> lst = [ 2, 4, 7, 9, 10, 12, 14, 17, 20 ]
>>> BinarySearchRecursive2( lst, 21 )
-10
>>> BinarySearchRecursive2( lst, 11 )
-6
>>> BinarySearchRecursive2( lst, 10 )
4
```


Why Helper Functions

Typically, we need a helper function because we need some extra information in the recursive calls. But, that information isn't needed for the "public call" to the function.

For example: For BinarySearchHelper, you need the low and high parameters to tell you which part of the list you're searching in this particular recursive call.

But for the top level call to BinarySearch, those parameters would always be 0 and len(lst). So why have those parameters on the top level function.

Let's Take a Break

Recursion vs. Iteration

For some problems a recursive solution is simpler to code and to understand than an *iterative* solution.

You can *always* convert a recursive solution to an iterative solution, and vice versa. But it may not be easy!

```
def gcdIter( a, b ):
    """ Iterative version of GCD. """
    while a != b:
        if a < b:
            b = b - a
        elif b < a:
            a = a - b
    return a
```

The Overhead of Recursion

Though recursion is a wonderful conceptual tool, *it's not free.* There is a cost to any recursive implementation.

Consider the computation of the nth Fibonacci number.

$$F(0) = 0$$
$$F(1) = 1$$
$$F(n) = F(n-1) + F(n-2), \text{ for } n \geq 2$$

Some of the Fibonacci Numbers:

n	0	1	2	3	4	5	6	7	8	9	10	11
$F(n)$	0	1	1	2	3	5	8	13	21	34	55	89

BTW: often the sequence is started at 1 rather than at 0.

Fibonacci in Python

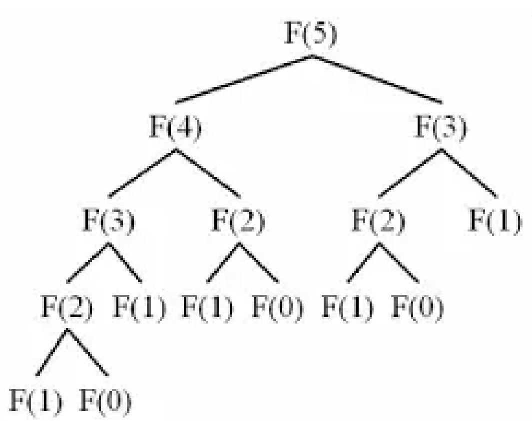
$$F(0) = 0$$
$$F(1) = 1$$
$$F(n) = F(n-1) + F(n-2), \text{ for } n \geq 2$$

```
def fib(n):
    """ Return nth Fibonacci number. """
    if n == 0:
        return 0
    elif n == 1:
        return 1
    else:
        return fib(n-1) + fib(n-2)
```

This is a very nice transcription of the recurrence relation and works fine. Sort of. What's wrong and how would you fix it?

How Bad Is It?

If n is 0 or 1, you only make one call to `fib`. But suppose $n = 5$, you do a lot of work, much of it repeated multiple times.



How many recursive calls are made?

Counting Calls

How many calls to fib are made for a given n?

The recurrence relation for this is:

$$C(0) = 1$$
$$C(1) = 1$$
$$C(n) = 1 + C(n-1) + C(n-2), \text{ for } n \geq 2$$

We can easily write a Python function to compute this:

```
def fibCountCalls( n ):
    if n == 0:
        return 1
    elif n == 1:
        return 1
    else:
        return 1 + fibCountCalls(n-1) + fibCountCalls(n-2)
```

How Many Calls

```
>>> fibTallyCalls( 20 )
i:  0   fib(i):     0   calls:       1
i:  1   fib(i):     1   calls:       1
i:  2   fib(i):     1   calls:       3
i:  3   fib(i):     2   calls:       5
i:  4   fib(i):     3   calls:       9
i:  5   fib(i):     5   calls:      15
i:  6   fib(i):     8   calls:      25
i:  7   fib(i):    13   calls:      41
i:  8   fib(i):    21   calls:      67
i:  9   fib(i):    34   calls:     109
i: 10   fib(i):    55   calls:     177
i: 11   fib(i):    89   calls:     287
i: 12   fib(i):   144   calls:     465
i: 13   fib(i):   233   calls:     753
i: 14   fib(i):   377   calls:    1219
i: 15   fib(i):   610   calls:    1973
i: 16   fib(i):   987   calls:    3193
i: 17   fib(i):  1597   calls:    5167
i: 18   fib(i):  2584   calls:    8361
i: 19   fib(i):  4181   calls:   13529
```


Instrumenting the Code

This computes `fib(n)` and keeps track of how long the computation takes and how many recursive calls are made.

```
import time

def fibCaller():
  while True:
    n = int(input("Input an integer (negative to exit): "))
    if n < 0:
        break
    # Time the call
    tStart = time.clock(); ans = fib(n); tEnd = time.clock()
    interval = tEnd - tStart
    intervalStr = format(interval, "9.4f")
    # How many recursive calls?
    calls = fibCountCalls( n )
    print ("fib(" + str(n) +") = " + str(ans), end = "")
    print (" with " + str( calls ) + " recursive calls")
    print ("time = " + intervalStr + " seconds to execute")
```


The Computation

You can see that the values go up quickly.

```
>>> fibCaller()
Input an integer (negative to exit): 10
fib(10) = 55 with 177 recursive calls
time =    0.0000 seconds to execute
Input an integer (negative to exit): 20
fib(20) = 6765 with 21891 recursive calls
time =    0.0053 seconds to execute
Input an integer (negative to exit): 30
fib(30) = 832040 with 2692537 recursive calls
time =    0.2702 seconds to execute
Input an integer (negative to exit): 40
fib(40) = 102334155 with 331160281 recursive calls
time =   31.0691 seconds to execute
 < I tried it for 50 but got tired of waiting >
Input an integer (negative to exit): -10
```


Can We Do Better?

Take another look at the initial values of the Fibonacci sequence:

n	0	1	2	3	4	5	6	7	8	9	10	11
$F(n)$	0	1	1	2	3	5	8	13	21	34	55	89

Surely we can do better than our horrible exponential solution. Perhaps instead of computing backwards from n down to 0, we can *compute forwards from 0 to n*.

A Better Implementation

n	0	1	2	3	4	5	6	7	8	9	10	11
$F(n)$	0	1	1	2	3	5	8	13	21	34	55	89

```
def fibHelper( k, limit, ans, ansSub1 ):
    if k >= limit:
        return ans
    else:
        return fibHelper( k+1, limit, ans + ansSub1, ans )

def fibBetter(n):
    return fibHelper( 1, n, 1, 0 )
```

Why was the `fibHelper` function needed?

Better Performance

After changing `fibCaller` to call `fibBetter`:

```
>>> fibBetterCaller()
Input an integer (negative to exit): 10
fib(10) = 55 with 10 recursive calls
time =    0.0000 seconds to execute
Input an integer (negative to exit): 20
fib(20) = 6765 with 20 recursive calls
time =    0.0000 seconds to execute
Input an integer (negative to exit): 30
fib(30) = 832040 with 30 recursive calls
time =    0.0000 seconds to execute
Input an integer (negative to exit): 40
fib(40) = 102334155 with 40 recursive calls
time =    0.0000 seconds to execute
Input an integer (negative to exit): 500
fib(500) =
    13942322456169788013972438287040728395007025658769730 7
26410896294832557162286329069155765887622252129412 5 with 500
     recursive calls
time =    0.0004 seconds to execute
Input an integer (negative to exit): -10
>>>
```


Better Performance

Is there any limit to how big an argument we can give? Yes, because the runtime stack will overflow when we reach the "recursion depth."

```
>>> fibBetterCaller()
Input an integer (negative to exit): 900
fib(900) =
    54877108839480000051413673948383714443800519309123592 72
44949534270398112010643412349543875215253906155049490921 87
44121824667910473144247302201398016040700701717569731790 04
83275246652938800 with 900 recursive calls
time =    0.0007 seconds to execute
Input an integer (negative to exit): 1000
Traceback (most recent call last):
  File "<stdin>", line 1, in <module>
< some lines omitted >
  [Previous line repeated 993 more times]
  File "/u/byoung/cs303e/slides/RecursionExamples.py", line
      73, in fibHelper
    if k >= limit:
RecursionError: maximum recursion depth exceeded in
    comparison
```


Iterative Version

You can replace the recursive code by *iterative* code (i.e., a loop) and you won't have this problem. Try this as an exercise.

```
>>> iterativeFib( 5000 )
387896845438832563370191630832590531208212771464624510616059
721489555013904403709701082291646221066947929345285888297381
348310200895498294036143015691147893836421656394410691021450
563413370655865623825465670071252592990385493381392883637834
751890876297071203333705292310769300851809384980180384781399
674888176555465378829164426891298038461377896902150229308247
566634622492307188332480328037503913035290330450584270114763
524227021093463769910400671417488329842289149127310405432875
329804427367682297724498774987455569190770388063704683279481
135897373999311010621930814901857081539785437919530561751076
105307568878376603366735544525884488624161921055345749367589
784902798823435102359984466393485325641195222185956306047536
464547076033090242080638258492915645287629157575914234380914
230291749108898415520985443248659407979357131684169286803954
530954538869811466508206686289742063932343848846524098874239
587380197699382031717420893226546887936400263079778005875912
967138963421425257911687275560036031137054775472460463998758
804698517840867438286312 5
```

Closed Form Solution

It turns out that there is a *closed form* solution for the nth Fibonacci number.

$$fib(n) = (1/\sqrt{5})[(1+\sqrt{5})/2]^n - (1/\sqrt{5})[(1-\sqrt{5})/2]^n.$$

So you don't even have to loop!

Let's Take a Break

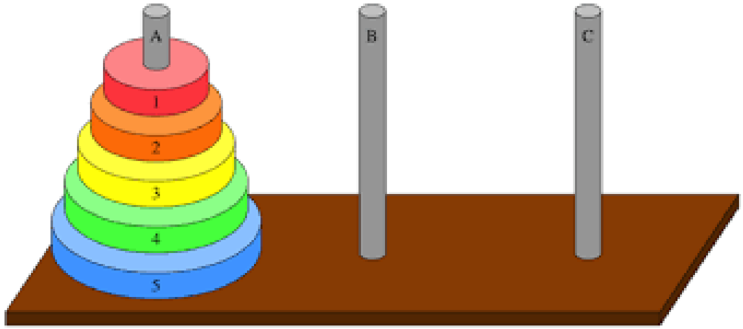
Naturally Recursive Problems

Some problems (like Fibonacci) have a very natural recursive solution, but can easily be recoded in a non-recursive fashion.

Some other problems are very difficult to solve in any way but recursively.

Towers of Hanoi

The *Towers of Hanoi* consists of three pegs and a number of disks of different sizes that can slide onto any peg. The puzzle starts with the disks neatly stacked in order of size on one peg, the smallest at the top.



Towers of Hanoi

The objective of the puzzle is to move the entire stack to another peg, obeying the following rules:

- Only one disk may be moved at a time.
- No disk may be placed on top of a smaller disk

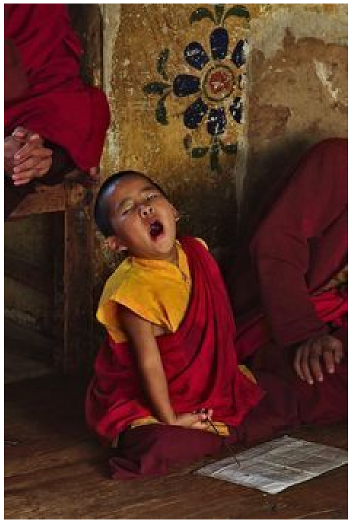
This is a problem which is easy to solve recursively, but very hard to solve iteratively.

The Legend

There is a legend about a Vietnamese temple which contains a large room with three time-worn posts in it holding 64 golden disks. The priests of Hanoi, acting out the command of an ancient prophecy, have been moving these disks, in accordance with the rules of the puzzle, for centuries.

According to the legend, when the last move of the puzzle is completed, the world will end. Are we in danger?

The Legend



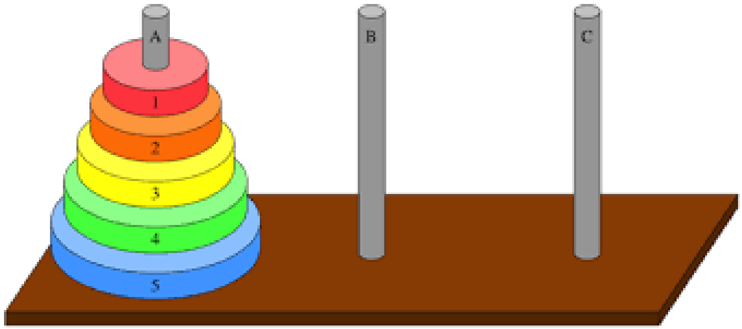
The smallest solution for 64 disks requires: 18,446,744,073,709,551,615 moves.

If the priests were able to move disks at a rate of one per second, using the smallest number of moves, it would take them $2^{64} - 1$ seconds or roughly 600 billion years.

Thinking Recursively

Problem: Move n disks from peg A to peg C, using peg B as the intermediate peg.

1. What's the base case?
2. What are the recursive cases?



Towers of Hanoi

What problem are we solving? Solving the puzzle means moving the disks from the start state to the ending state, while following the rules.

Our program is solving a slightly different problem: print out a legal list of moves that will take us from start state to end state.

So, what's a move?

```
def makeMove(peg1, peg2):
    print ("Move disk from " + peg1 + " to " + peg2)
```


Towers of Hanoi

Our main function has four parameters:

n : how many disks to move (int);
A : the start peg (str);
B : the intermediate peg (str);
C : the destination peg (str).

```
def towersOfHanoi(n, A, B, C):
    """ Prints a list of legal moves to solve
        the Tower of Hanoi problem for n disks. """
    if n == 1:
        makeMove(A, C)
    else:
        towersOfHanoi(n-1, A, C, B)
        makeMove(A, C)
        towersOfHanoi(n-1, B, A, C)
```


Calling It

```
>>> towersOfHanoi( 4, 'A', 'B', 'C' )
  Move disk from A to B     #  \
  Move disk from A to C     #   \
  Move disk from B to C     #    \
  Move disk from A to B     #  Hanoi(3, A, C, B)
  Move disk from C to A     #    /
  Move disk from C to B     #   /
  Move disk from A to B     #  /
  Move disk from A to C     #  move (A, C )
  Move disk from B to C     #  \
  Move disk from B to A     #   \
  Move disk from C to A     #    \
  Move disk from B to C     #  Hanoi(3, B, A, C)
  Move disk from A to B     #    /
  Move disk from A to C     #   /
  Move disk from B to C     #  /
>>>
```


How Many Moves

```
def towersOfHanoi(n, frm, using, to):
    """ Prints a list of legal moves to solve
        the Tower of Hanoi problem for n disks. """
    if n == 1:
        makeMove(frm, to)
    else:
        towersOfHanoi(n-1, frm, to, using)
        makeMove(frm, to)
        towersOfHanoi(n-1, using, frm, to)
```

It's pretty clear that the number of moves (and recursive calls) is defined by:

$$M(1) = 1$$
$$M(n) = 1 + 2 * M(n-1), \text{ for } n > 1$$

But this has solution:

$$M(n) = 2^n - 1$$

How Many Calls?

The number of moves is also the number of recursive calls.

```
>>> for i in range(0, 64, 5):
...     print( format( i, "3d"), \
...            format(towersMoveCount(i), "20d" ))
...
  0                    0
  5                   31
 10                 1023
 15                32767
 20              1048575
 25             33554431
 30           1073741823
 35          34359738367
 40        1099511627775
 45       35184372088831
 50     1125899906842623
 55    36028797018963967
 60  1152921504606846975
>>>
```

If there are so many recursive calls, why doesn't the program crash?

Next stop: Turtle Graphics.