M 340L – CS
Homework Set 5

1. Either show that the vectors $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ are linearly independent or express one as a linear combination of the others.

$$\begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 3 \\ 3 & -8 & -1 \end{bmatrix} \to \begin{bmatrix} 3 & -8 & -1 \\ 2 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & -8 & -1 \\ 0 & 16/3 & 11/3 \\ 0 & 0 & 1 \end{bmatrix}$$. The vectors are linearly independent.

2. Either show that the columns of the matrix $\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 5 \\ 1 & 2 & -5 \\ 2 & 1 & -10 \end{bmatrix}$ are linearly independent or find a solution to the homogeneous problem.

$$\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 5 \\ 1 & 2 & -5 \\ 2 & 1 & -10 \end{bmatrix} \to \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 5 \\ 0 & 5/4 & -5 \\ 0 & -1/2 & -10 \end{bmatrix} \to \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 5 \\ 0 & 5/4 & -5 \\ 0 & -1/2 & -10 \end{bmatrix} \to \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 5 \\ 0 & 0 & 5/4 \\ 0 & 0 & -25/2 \end{bmatrix}$$

$$\to \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 5 \\ 0 & 0 & 5/4 \\ 0 & 0 & 0 \end{bmatrix}$$

The columns of the matrix are linearly independent.

3. Either show that the columns of the matrix $\begin{bmatrix} 1 & -2 & 3 & 2 \\ -2 & 4 & -6 & 2 \\ 0 & 1 & -1 & 3 \end{bmatrix}$ are linearly independent or find a solution to the homogeneous problem.

$\begin{bmatrix} 1 & -2 & 3 & 2 \\ -2 & 4 & -6 & 2 \\ 0 & 1 & -1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 0 & 0 & 6 \\ 0 & 1 & -1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 6 \end{bmatrix}$. The columns of the matrix are not linearly independent and a solution to the homogeneous problem is

$x_4 = \frac{0}{6} = 0,\;\; x_3$ is free – we take

$x_3 = 1, x_2 = \frac{0-(-1)\cdot 1 - 3\cdot 0}{1} = 1, x_1 = \frac{0-(-2)\cdot 1 - 3\cdot 1 - 2\cdot 0}{1} = -1.$

4. Find the inverses of these matrices if they exist:

a. $\begin{bmatrix} 2 & 4 \\ 4 & 6 \end{bmatrix}$

We need to solve the systems $\begin{bmatrix} 2 & 4 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & 4 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for the columns of the inverse matrix $\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$. We transform

$\begin{bmatrix} 2 & 4 & 1 & 0 \\ 4 & 6 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{bmatrix}$ and solve

$x_{21} = \frac{-2}{-2} = 1, x_{11} = \frac{1-4\cdot 1}{2} = -3/2, x_{22} = \frac{1}{-2} = -1/2, x_{11} = \frac{0-4\cdot(-1/2)}{2} = 1.$ We get

$\begin{bmatrix} 2 & 4 \\ 4 & 6 \end{bmatrix}^{-1} = \begin{bmatrix} -3/2 & 1 \\ 1 & -1/2 \end{bmatrix}.$

b. $\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{bmatrix}$

We need to solve the systems $\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{bmatrix}\begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$,
and $\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{bmatrix}\begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ for the columns of the inverse matrix $\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$.
We transform
$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ -4 & -7 & 3 & 0 & 1 & 0 \\ -2 & -6 & 4 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 10 & 2 & 1 \end{bmatrix}$ and see
that the system matrix has linearly dependent columns and thus no inverse exists.

5. Use the inverse found in Exercise 4a to solve the system
$$\begin{bmatrix} 2 & 4 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We have $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3/2 & 1 \\ 1 & -1/2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 3/2 \end{bmatrix}$.

6. Mark each statement True or False. Supply a **simple counterexample** for each false statement.

a. If A is invertible, then the inverse of A^{-1} is A itself.

True. $(A^{-1})^{-1} = A.$

b. A product of invertible $n \times n$ matrices is invertible, and the inverse of the product is the product of their inverses in the same order.

False. Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, then $AB = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ so $(AB)^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$ but

$A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, B^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$, so

$$A^{-1}B^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \neq \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}.$$