M 340L – CS
Homework Set 10 Solutions

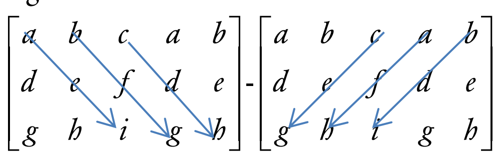
1. Calculate the determinants of

a. $\begin{bmatrix} 3 & 6 \\ -1 & 4 \end{bmatrix}$, $\det(\begin{bmatrix} 3 & 6 \\ -1 & 4 \end{bmatrix}) = 3\cdot 4 - 6\cdot(-1) = 18.$

b. $\begin{bmatrix} 2 & 2 & 4 \\ -2 & 0 & 3 \\ 4 & 3 & -1 \end{bmatrix}$, $\det(\begin{bmatrix} 2 & 2 & 4 \\ -2 & 0 & 3 \\ 4 & 3 & -1 \end{bmatrix})$

$$= 2\cdot 0\cdot(-1) - 2\cdot 3\cdot 3 - (-2)\cdot 2\cdot(-1) + (-2)\cdot 3\cdot 4 + 4\cdot 2\cdot 3 - 4\cdot 0\cdot 4 = -22.$$

2. The expansion of a 3 x 3 determinant can be remembered by the following device. Add a copy of the first two columns to the right of the matrix, and compute the determinant adding the products along the northwest-to-southeast diagonals and subtracting the products along the northeast-to-southwest diagonals:



Use this method to compute the determinants:

a. $\begin{bmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{bmatrix}$, $\det(\begin{bmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{bmatrix})$

$$= 0\cdot(-3)\cdot 1 + 5\cdot 0\cdot 2 + 1\cdot 4\cdot 4 - 1\cdot(-3)\cdot 2 - 0\cdot 0\cdot 4 - 5\cdot 4\cdot 1 = 2.$$

b. $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix}$, $\det(\begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix})$

$$= 1\cdot 1\cdot 2 + 3\cdot 1\cdot 3 + 5\cdot 2\cdot 4 - 5\cdot 1\cdot 3 - 1\cdot 1\cdot 4 - 3\cdot 2\cdot 2 = 20.$$

3. Prove that for an invertible matrix A, $\det(A^{-1}) = 1/\det(A)$. (Hint: Remember $AA^{-1} = I$.)

$$1 = \det(I) = \det(AA^{-1}) = \det(A)\cdot\det(A^{-1}), \text{ so } \det(A^{-1}) = 1/\det(A).$$

4. Answer true or false to the following. If false offer a counterexample.

a. If the columns of A are linearly dependent, then $\det(A)=0$.

True. If the columns of A are linearly dependent, then A is singular and $\det(A)=0$.

b. $\det(A+B)=\det(A)\det(B)$.

False. Let $A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, B=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$ then

$\det(A+B)=\det(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix})=1\neq 0=0\cdot 0=\det(A)\det(B)$.

c. The determinant of A is the product of the diagonal entries in A.

False. Let $A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, then $\det(A)=-1\neq 0\cdot 0$.

d. If $\det(A)$ is zero, then two rows or two columns are the same, or a row or a column is zero.

False. Let $A=\begin{bmatrix}2 & 4\\ 1 & 2\end{bmatrix}$, then $\det(A)=0$ but no two rows nor two columns are the same, nor is a row or a column zero.

5. Answer true or false to the following. If false offer a counterexample.

a. If $Ax=\lambda x$ for some scalar λ, then x is an eigenvector of A.

False. Let $A=[1], x=[0]$ then $Ax=0=0\cdot 0=0x$ but $x=[0]$ is not an eigenvector of A.

b. If $v1$ and $v2$ are linearly independent eigenvectors, then they correspond to distinct eigenvalues.

False. Let $A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, v1=\begin{bmatrix}1\\ 0\end{bmatrix}, v2=\begin{bmatrix}0\\ 1\end{bmatrix}$ then $A\,v1=1v1$ and $A\,v2=1v2$, so both $v1$ and $v2$ are linearly independent eigenvectors with the common eigenvalue 1.

c. The eigenvalues of a matrix are on its main diagonal.

False. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $v1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ then $A\,v1 = 1v1$ and $A\,v2 = -1v2$, so the eigenvalues are 1 and -1, neither ow which is on the diagonal.

6. For each of these matrices,

- find the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.
- factor it to get the eigenvalues: $\lambda_1, \lambda_2, \ldots, \lambda_n$.
- for i= $1, \ldots, n$: find x^i the eigenvector corresponding λ_i.(that is, find a vector x^i in the nullspace of $A - \lambda_i I$).
- Scale all eigenvectors so the largest component is + 1.

a. $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$.

The characteristic polynomial is $(1-\lambda)(2-\lambda) - 12 = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2)$ so the eigenvalues are 5 and -2. The null space of $A - 5I = \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix}$ is the vector $\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$ (and its multiples). The null space of $A - (-2)I = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}$ is the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (and its multiples). Thus the eigenvectors are $\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

b. $A=\begin{bmatrix} 8 & -12 & 0 \\ -3 & 9 & -3 \\ -5 & 3 & 3 \end{bmatrix}$.

The characteristic polynomial is
$(8-\lambda)(9-\lambda)(3-\lambda)+(-12)(-3)(-5)+0-0-(8-\lambda)(-3)3-(3-\lambda)(-3)(-12)$
$=-\lambda^3+20\lambda^2-\lambda 96=-(\lambda-8)(\lambda-12)(\lambda-0)$
so the eigenvalues are 8, 12, and 0. The null space of $A-8I=\begin{bmatrix} 0 & -12 & 0 \\ -3 & 1 & -3 \\ -5 & 3 & -5 \end{bmatrix}$ is the vector $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ (and its multiples) since
$\begin{bmatrix} 0 & -12 & 0 \\ -3 & 1 & -3 \\ -5 & 3 & -5 \end{bmatrix} \to \begin{bmatrix} -5 & 3 & -5 \\ -3 & 1 & -3 \\ 0 & -12 & 0 \end{bmatrix} \to \begin{bmatrix} -5 & 3 & -5 \\ 0 & -4/5 & 0 \\ 0 & -12 & 0 \end{bmatrix} \to \begin{bmatrix} -5 & 3 & -5 \\ 0 & -4/5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The null space of $A-12I=\begin{bmatrix} -4 & -12 & 0 \\ -3 & -3 & -3 \\ -5 & 3 & -9 \end{bmatrix}$ is the vector $\begin{bmatrix} 1 \\ -1/3 \\ -2/3 \end{bmatrix}$ (and its multiples) since
$\begin{bmatrix} -4 & -12 & 0 \\ -3 & -3 & -3 \\ -5 & 3 & -9 \end{bmatrix} \to \begin{bmatrix} -4 & -12 & 0 \\ 0 & 6 & -3 \\ 0 & 18 & -9 \end{bmatrix} \to \begin{bmatrix} -4 & -12 & 0 \\ 0 & 6 & -3 \\ 0 & 0 & 0 \end{bmatrix}$. The null space of
$A-0I=\begin{bmatrix} 8 & -12 & 0 \\ -3 & 9 & -3 \\ -5 & 3 & 3 \end{bmatrix}$ is the vector $\begin{bmatrix} 1 \\ 2/3 \\ 1 \end{bmatrix}$ (and its multiples) since
$\begin{bmatrix} 8 & -12 & 0 \\ -3 & 9 & -3 \\ -5 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 8 & -12 & 0 \\ 0 & 9/2 & -3 \\ 0 & -9/2 & 3 \end{bmatrix} \to \begin{bmatrix} 8 & -12 & 0 \\ 0 & 9/2 & -3 \\ 0 & 0 & 0 \end{bmatrix}$. Thus the eigenvectors are $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$,
$\begin{bmatrix} 1 \\ -1/3 \\ -2/3 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 2/3 \\ 1 \end{bmatrix}$.

7. Two eigenvectors of an upper triangular matrix:

Let $U=\begin{bmatrix} a & b & \cdots \\ 0 & c & \cdots \\ 0 & 0 & \ddots \end{bmatrix}$ be n by n and upper triangular. Assume $a \neq c$.

a. Show that the eigenvector corresponding to the eigenvalue a is e_1 (i.e the first column of the n by n identity matrix). (Use this below.)

$$Ue_1=\begin{bmatrix} a & b & \cdots \\ 0 & c & \cdots \\ 0 & 0 & \ddots \end{bmatrix} e_1=\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}=ae_1.$$

b. Show that the eigenvector corresponding to the eigenvalue c is $\begin{bmatrix} b/(c-a) \\ 1 \\ 0 \\ \vdots \end{bmatrix}$. (Use this or a scaled version $\begin{bmatrix} 1 \\ (c-a)/b \\ 0 \\ \vdots \end{bmatrix}$. below.)

$$U\begin{bmatrix} b/(c-a) \\ 1 \\ 0 \\ \vdots \end{bmatrix}.=\begin{bmatrix} a & b & \cdots \\ 0 & c & \cdots \\ 0 & 0 & \ddots \end{bmatrix}\begin{bmatrix} b/(c-a) \\ 1 \\ 0 \\ \vdots \end{bmatrix}=\begin{bmatrix} ab/(c-a)+b \\ c \\ 0 \end{bmatrix}=c\begin{bmatrix} b/(c-a) \\ 1 \\ 0 \\ \vdots \end{bmatrix}.$$