M 340L – CS
Homework Set 10 Solutions

1. Let $u^1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, u^2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, u^3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, b = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$.

a. Form the matrix $U = \begin{bmatrix} u^1 & u^2 & u^3 \end{bmatrix}$ and confirm that the columns of U are orthogonal by computing $U^T U$.

$U = \begin{bmatrix} 3 & 2 & 1 \\ -3 & 2 & 1 \\ 0 & -1 & 4 \end{bmatrix}, U^T U = \begin{bmatrix} 3 & -3 & 0 \\ 2 & 2 & -1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ -3 & 2 & 1 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 18 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 18 \end{bmatrix}$ which is diagonal so the columns of U are orthogonal.

b. Express b as a linear combination of u^1, u^2 and u^3. (That is, solve $Ux = b$.)

If $Ux = b$, then $U^T U x = U^T b$, so $\begin{bmatrix} 18 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 18 \end{bmatrix} x = \begin{bmatrix} 3 & -3 & 0 \\ 2 & 2 & -1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 24 \\ 3 \\ 6 \end{bmatrix}$, so

$$x = \begin{bmatrix} 24/18 \\ 3/9 \\ 6/18 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 1/3 \\ 1/3 \end{bmatrix}.$$

2. Let $u^1 = \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, u^2 = \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix}$

a. Form the matrix $U = \begin{bmatrix} u^1 & u^2 \end{bmatrix}$ and confirm that the columns of U are orthogonal by computing $U^T U$.

$U = \begin{bmatrix} -2/3 & 1/3 \\ 1/3 & 2/3 \\ 2/3 & 0 \end{bmatrix}, U^T U = \begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 1/3 & 2/3 & 0 \end{bmatrix} \begin{bmatrix} -2/3 & 1/3 \\ 1/3 & 2/3 \\ 2/3 & 0 \end{bmatrix} = \begin{bmatrix} 10/9 & 0 \\ 0 & 5/9 \end{bmatrix}$ which is diagonal so the columns of U are orthogonal.

b. Normalize the columns and confirm that $U^T U = I$.

After normalization,

$$U = \begin{bmatrix} -2/\sqrt{10} & 1/\sqrt{5} \\ 1/\sqrt{10} & 2/\sqrt{5} \\ 2/\sqrt{10} & 0 \end{bmatrix}, UTU = \begin{bmatrix} -2/\sqrt{10} & 1/\sqrt{10} & 2/\sqrt{10} \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} -2/\sqrt{10} & 1/\sqrt{5} \\ 1/\sqrt{10} & 2/\sqrt{5} \\ 2/\sqrt{10} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

3. Answer true or false to the following. If false offer a counterexample.

a. Every orthogonal set in $\mathbb{R}^n$ is linearly independent.

False. The set consisting of a zero vector alone is orthogonal but not linearly independent.

b. If a set $S = \{u^1, u^2, ..., u^k\}$ has the property that $u^i \cdot u^j = 0$ whenever $i \neq j$, then S is an orthonormal set.

False. The set consisting of a zero vector alone has the property that $u^i \cdot u^j = 0$ whenever $i \neq j$, but S is not an orthonormal set.

4. Show that if U is a square orthogonal matrix then $U^T = U^{-1}$.

We have $U^T U = I$, so $U^T = U^{-1}$.

5. Show that if U is an $m \times n$ orthogonal matrix then for all $x \in \mathbb{R}^n, \|Ux\| = \|x\|$. (This can be stated as "An orthogonal transformation preserves length.".)

For all $x \in \mathbb{R}^n, \|Ux\|^2 = (Ux)^T (Ux) = x^T U^T U x = x^T x = \|x\|^2$.

6. Show that if P is a projection then so is $I - P$. (Remember the definition of a projection.)

Since P is a projection, $P^2 = P$, but then
$(I-P)^2 = (I-P)(I-P) = I - P - P + P^2 = I - P - P + P = I - P$, so $I - P$ is also a projection.

7. Consider this mathematical (and not necessarily computer) procedure:

$[\alpha, v'] =$ **project** $[u, v]$

Inputs vectors u and v, computes and Returns $\alpha = u \cdot v / u \cdot u$ and $v' = v - \alpha u$.

Now, let $u^1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, u^2 = \begin{bmatrix} -1 \\ 5 \\ -1 \end{bmatrix}, u^3 = \begin{bmatrix} 9 \\ -3 \\ 3 \end{bmatrix}$.

a. $[r_{1,2}, u_2'] =$ **project** $[u_1, u_2]$. (That is, subtract the projection of u_2 onto the subspace spanned by u_1.)

project $[\begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -1 \end{bmatrix}]$ yields $r_{1,2} = -18/18 = -1$, and $u_2' = \begin{bmatrix} -1 \\ 5 \\ -1 \end{bmatrix} - (-1)\begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$.

b. $[r_{1,3}, u_3'] =$ **project** $[u_1, u_3]$. (That is, subtract the projection of u_3 onto the subspace spanned by u_1.)

project $[\begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ -3 \\ 3 \end{bmatrix}]$ yields $r_{1,3} = 36/18 = 2$, and $u_3' = \begin{bmatrix} 9 \\ -3 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$.

c. $[r_{2,3}, u_3''] =$ **project** $[u_2', u_3']$. (That is, subtract the projection of u_3' onto the subspace spanned by u_2'.)

project $[\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}]$ yields $r_{2,3} = 9/9 = 1$, and $u_3' = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} - 1\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$.

d. Compute $A = \begin{bmatrix} u_1 & u_2' & u_3'' \end{bmatrix} \begin{bmatrix} 1 & r_{1,2} & r_{1,3} \\ 0 & 1 & r_{2,3} \\ 0 & 0 & 1 \end{bmatrix}$. (Compare A to $\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$ and compare $\begin{bmatrix} u_1 & u_2' & u_3'' \end{bmatrix}$ to U in Problem 1. You have just used the Gram-Schmidt Algorithm to orthogonalize vectors.)

$$A = \begin{bmatrix} 3 & 2 & 1 \\ -3 & 2 & 1 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 9 \\ -3 & 5 & -3 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \text{ and}$$

$$\begin{bmatrix} u_1 & u_2' & u_3'' \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ -3 & 2 & 1 \\ 0 & -1 & 4 \end{bmatrix} = U \text{ in Problem 1.}$$