

**M 340L - CS**  
**Homework Set 6 Solutions**

1. Suppose  $P$  is invertible and  $A = PBP^{-1}$ . Solve for  $B$  in terms of  $P$  and  $A$ .

$$\text{Since } A = PBP^{-1}, \text{ we have } B = P^{-1}(PBP^{-1})P = P^{-1}AP.$$

2. Suppose  $(B - C)D = 0$ , where  $B$  and  $C$  are  $m \times n$  matrices and  $D$  is invertible. Prove that  $B = C$ .

$$\begin{aligned} \text{We have } (B - C)D = 0, \text{ so } B - C &= (B - C)DD^{-1} = 0D^{-1} = 0 \text{ and} \\ B &= C + (B - C) = C + 0 = C. \end{aligned}$$

3. Suppose  $A$  and  $B$  are square matrices,  $B$  is invertible, and  $AB$  is invertible. Prove that  $A$  is invertible. [Hint: Let  $C = AB$ , and solve this equation for  $A$  in terms of  $B$  and  $C$ .]

$$\text{If } C = AB, \text{ we have } A = ABB^{-1} = CB^{-1}, \text{ so } A^{-1} = (CB^{-1})^{-1} = (B^{-1})^{-1}C^{-1} = B(AB)^{-1}.$$

4. Solve the equation  $AB = BC$  for  $A$ , assuming that  $A$ ,  $B$ , and  $C$  are square and  $B$  is invertible.

$$\text{We have } A = ABB^{-1} = BCB^{-1}.$$

5. Answer true or false to the following. If false offer a counterexample.

a. If  $u$  and  $v$  are linearly independent, and if  $w$  is in  $\text{Span}\{u, v\}$ , then  $u, v, w$  are linearly dependent.

**True.** If  $w$  is in  $\text{Span}\{u, v\}$  it must be a linear combination of  $u$  and  $v$ .

b. If three vectors in  $\mathbb{R}^3$  lie in the same plane in  $\mathbb{R}^3$ , then they are linearly dependent.

**False.** The three vectors  $u = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  are in  $\mathbb{R}^3$  lie and lie in the plane

$x_3 = 1$ , but are linearly independent.

c. If a set contains fewer vectors than there are entries in the vectors, then the set is linearly independent.

**False.** The set of the single vector  $u = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  has two entries but is not linearly independent.

d. If a set in  $\mathbb{R}^n$  is linearly dependent then the set contains more than  $n$  vectors.

**False.** The set of the single vector  $u = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathbb{R}^2$  is linearly dependent but does not contain more than 2 vectors.

e. If  $v_1$  and  $v_2$  are in  $\mathbb{R}^4$  and  $v_2$  is not a scalar multiple of  $v_1$ , then  $v_1, v_2$  are linearly independent.

**False.** With the vectors  $v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^4$ ,  $v_2$  is not a scalar multiple of  $v_1$ , but  $v_1, v_2$  are linearly dependent since  $v_1 = 0v_2$ .

f. If  $v_1, v_2, v_3$  are in  $\mathbb{R}^3$  and  $v_3$  is not a linear combination of  $v_1, v_2$ , then  $v_1, v_2, v_3$  are linearly independent.

**False.** With the vectors  $v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \mathbb{R}^3$ ,  $v_3$  is not a linear combination of  $v_1, v_2$ , but  $v_1, v_2, v_3$  are linearly dependent since  $v_1 = 0v_2 + 0v_3$ .

g. If  $\{v_1, v_2, v_3, v_4\}$  is a linearly independent set of vectors in  $\mathbb{R}^4$ , then  $\{v_1, v_2, v_3\}$  is also linearly independent.

**True.** If no linear combination of the elements of  $\{v_1, v_2, v_3, v_4\}$  is zero then no linear combination of the elements of  $\{v_1, v_2, v_3\}$  is zero.

6. Answer true or false to the following. If false offer a counterexample.

a. The range of the transformation  $x \mapsto Ax$  is the set of all linear combinations of the columns of  $A$ .

**True.** The range of the transformation  $x \mapsto Ax$  is the set of all vectors of the form  $Ax$  and that equals the set of all linear combinations of the columns of  $A$ .

b. Every matrix transformation is a linear transformation.

**True.** We have  $A(\alpha x) = \alpha Ax$  and  $A(x + y) = Ax + Ay$  so the transformation  $x \mapsto Ax$  is linear.

c. A linear transformation preserves the operations of vector addition and scalar multiplication.

**True.** We have  $T(\alpha x) = A(\alpha x) = \alpha Ax = \alpha T(x)$  and  $T(x + y) = A(x + y) = Ax + Ay = T(x) + T(y)$  so the operations of vector addition and scalar multiplication are preserved.

d. A linear transformation  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  always maps the origin of  $\mathbb{R}^n$  to the origin of  $\mathbb{R}^m$ .

**True.** Since  $T(0) = T(0x) = 0T(x) = 0$  for any vector  $x$ , a linear transformation  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  always maps the origin of  $\mathbb{R}^n$  to the origin of  $\mathbb{R}^m$ .

7. Find formulas for  $X$ ,  $Y$ , and  $Z$  in terms of  $I$ ,  $A$ ,  $B$ , and  $C$  and inverses. Assume  $A$ ,  $B$ , and  $C$  have inverses. (Hint: Compute the product on the left, and set it equal to the right side. First, pretend the blocks are simply real numbers but make sure you do not ever divide - you may multiply by inverses, however. Be careful about right and left multiplication.) In all cases, assume the block matrix dimensions are such that the products are defined.

a. 
$$\begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix} \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

We have  $XA + 0B = I$ ,  $YA + ZB = 0$ ,  $X0 + 0C = 0$ , and  $Y0 + ZC = I$ , so

$XA = I$ ,  $YA = -ZB$ , and  $ZC = I$ . We conclude that

$X = A^{-1}$ ,  $Z = C^{-1}$ , and  $Y = -ZBA^{-1} = -C^{-1}BA^{-1}$ .

$$\text{b. } \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} X & Y & Z \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

We have

$AX + B0 = I, 0X + I0 = 0, AY + B0 = 0, 0Y + I0 = 0, AZ + BI = 0,$  and  $0Z + II = I,$  so  
 $AX = I, AY = 0,$  and  $AZ = -B.$  We conclude that  $X = A^{-1}, Y = 0,$  and  $Z = -A^{-1}B.$

$$\text{c. } \begin{bmatrix} I & 0 & 0 \\ A & I & 0 \\ B & C & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ X & I & 0 \\ Y & Z & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

Omitting the obvious relations, We have

$AI + IX = 0, BI + CX + IY = 0,$  and  $B0 + CI + IZ = 0,$  so  
 $A + X = 0, B + CX + Y = 0,$  and  $C + Z = 0.$  We conclude that  
 $X = -A, Z = -C,$  and  $Y = -B - CX = CA - B.$

8. Answer true or false to the following. If false offer a counterexample.

a. If  $A$  is a  $4 \times 3$  matrix, then the transformation  $x \mapsto Ax$  maps  $\mathbb{R}^3$  onto  $\mathbb{R}^4$ .

**False.** Let  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  and  $b = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ , then for  $x \in \mathbb{R}^3$ ,  $Ax$  has zeros in components 2, 3, and 4 and thus for no  $x$  is  $Ax = b$ .

b. Every linear transformation from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  is a matrix transformation.

**True.** Let  $A$  be an  $m \times n$  matrix with columns  $T(e_1), \dots, T(e_n)$  (where  $e_1, \dots, e_n$  are the columns of  $I_n$ ), then the matrix transformation  $x \mapsto Ax$  is equivalent to  $x \mapsto T(x)$ .

c. The columns of the standard matrix for a linear transformation from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  are the images under  $T$  of the columns of the  $n \times n$  identity matrix.

**True.** Let  $A$  be an  $m \times n$  matrix with columns  $T(e_1), \dots, T(e_n)$  (where  $e_1, \dots, e_n$  are the columns of  $I_n$ ), then the matrix transformation  $x \mapsto Ax$  is equivalent to  $x \mapsto T(x)$ .

d. A mapping  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is one-to-one if each vector in  $\mathbb{R}^n$  maps onto a unique vector in  $\mathbb{R}^m$  (meaning two vectors in  $\mathbb{R}^n$  do not map to the same vector in  $\mathbb{R}^m$ ).

**True.** A function from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  is one-to-one if and only if two vectors in  $\mathbb{R}^n$  do not map to the same vector in  $\mathbb{R}^m$ .