M 340L – CS
Homework Set 12 Solutions

Note: Scale all eigenvectors so the largest component is +1.

1. For each of these matrices,

- find the characteristic polynomial $p(\lambda)=\det(A-\lambda I)$.
- factor it to get the eigenvalues: $\lambda_1,\lambda_2,...,\lambda_n$.
- for *i*= *1*,…, *n:* find x^i the eigenvector corresponding λ_i. (that is, find a vector x^i in the nullspace of $A-\lambda_i I$).

a. $A=\begin{bmatrix}1 & 3\\ 4 & 2\end{bmatrix}$.

The characteristic polynomial is $(1-\lambda)(2-\lambda)-12=\lambda^2-3\lambda-10=(\lambda-5)(\lambda+2)$ so the eigenvalues are 5 and -2. The null space of $A-5I=\begin{bmatrix}-4 & 3\\ 4 & -3\end{bmatrix}$ is the vector $\begin{bmatrix}3/4\\ 1\end{bmatrix}$ (and its multiples). The null space of $A-(-2)I=\begin{bmatrix}3 & 3\\ 4 & 4\end{bmatrix}$ is the vector $\begin{bmatrix}1\\ -1\end{bmatrix}$ (and its multiples). Thus the eigenvectors are $\begin{bmatrix}3/4\\ 1\end{bmatrix}$ and $\begin{bmatrix}1\\ -1\end{bmatrix}$.

b. $A=\begin{bmatrix}8 & -12 & 0\\ -3 & 9 & -3\\ -5 & 3 & 3\end{bmatrix}$.

The characteristic polynomial is

$$(8-\lambda)(9-\lambda)(3-\lambda)+(-12)(-3)(-5)+0-0-(8-\lambda)(-3)3-(3-\lambda)(-3)(-12)$$
$$=-\lambda^3+20\lambda^2-\lambda 96=-(\lambda-8)(\lambda-12)(\lambda-0)$$

so the eigenvalues are 8, 12, and 0. The null space of $A-8I=\begin{bmatrix}0 & -12 & 0\\ -3 & 1 & -3\\ -5 & 3 & -5\end{bmatrix}$ is the vector $\begin{bmatrix}1\\ 0\\ -1\end{bmatrix}$ (and its multiples) since

$$\begin{bmatrix}0 & -12 & 0\\ -3 & 1 & -3\\ -5 & 3 & -5\end{bmatrix}\rightarrow\begin{bmatrix}-5 & 3 & -5\\ -3 & 1 & -3\\ 0 & -12 & 0\end{bmatrix}\rightarrow\begin{bmatrix}-5 & 3 & -5\\ 0 & -4/5 & 0\\ 0 & -12 & 0\end{bmatrix}\rightarrow\begin{bmatrix}-5 & 3 & -5\\ 0 & -4/5 & 0\\ 0 & 0 & 0\end{bmatrix}$$. The null

space of $A-12I = \begin{bmatrix} -4 & -12 & 0 \\ -3 & -3 & -3 \\ -5 & 3 & -9 \end{bmatrix}$ is the vector $\begin{bmatrix} 1 \\ -1/3 \\ -2/3 \end{bmatrix}$ (and its multiples) since

$\begin{bmatrix} -4 & -12 & 0 \\ -3 & -3 & -3 \\ -5 & 3 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} -4 & -12 & 0 \\ 0 & 6 & -3 \\ 0 & 18 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} -4 & -12 & 0 \\ 0 & 6 & -3 \\ 0 & 0 & 0 \end{bmatrix}$. The null space of

$A-0I = \begin{bmatrix} 8 & -12 & 0 \\ -3 & 9 & -3 \\ -5 & 3 & 3 \end{bmatrix}$ is the vector $\begin{bmatrix} 1 \\ 2/3 \\ 1 \end{bmatrix}$ (and its multiples) since

$\begin{bmatrix} 8 & -12 & 0 \\ -3 & 9 & -3 \\ -5 & 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 8 & -12 & 0 \\ 0 & 9/2 & -3 \\ 0 & -9/2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 8 & -12 & 0 \\ 0 & 9/2 & -3 \\ 0 & 0 & 0 \end{bmatrix}$. Thus the eigenvectors are $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$,

$\begin{bmatrix} 1 \\ -1/3 \\ -2/3 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 2/3 \\ 1 \end{bmatrix}$.

2. Two eigenvectors of an upper triangular matrix:

Let $U = \begin{bmatrix} a & b & \cdots \\ 0 & c & \cdots \\ 0 & 0 & \ddots \end{bmatrix}$ be *n* by *n* and upper triangular. Assume $a \neq c$.

a. Show that the eigenvector corresponding to the eigenvalue a is e_1 (i.e the first column of the *n* by *n* identity matrix). (Use this below.)

$$Ue_1 = \begin{bmatrix} a & b & \cdots \\ 0 & c & \cdots \\ 0 & 0 & \ddots \end{bmatrix} e_1 = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} = ae_1.$$

b. Show that the eigenvector corresponding to the eigenvalue c is $\begin{bmatrix} b/(c-a) \\ 1 \\ 0 \\ \vdots \end{bmatrix}$. (Use this or a

scaled version $\begin{bmatrix} 1 \\ (c-a)/b \\ 0 \\ \vdots \end{bmatrix}$. below.)

$$U\begin{bmatrix} b/(c-a) \\ 1 \\ 0 \\ \vdots \end{bmatrix} .= \begin{bmatrix} a & b & \cdots \\ 0 & c & \cdots \\ 0 & 0 & \ddots \end{bmatrix} \begin{bmatrix} b/(c-a) \\ 1 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} ab/(c-a)+b \\ c \\ 0 \end{bmatrix} = c\begin{bmatrix} b/(c-a) \\ 1 \\ 0 \\ \vdots \end{bmatrix}.$$

3. How do perturbations affect eigenvalues and eigenvectors?

a. Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$. What are its eigenvalues and eigenvectors? (See the note above regarding the scaling of eigenvectors and make sure you do it throughout the homework.)

The eigenvalues are 2 and 2. The null space of $A-2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (and its multiples). $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is only one linearly independent eigenvector.

b. Let $B = \begin{bmatrix} 2 & 1 \\ 0 & 2+\varepsilon \end{bmatrix}$. What are its eigenvalues and eigenvectors? (Your answers should be in terms of the perturbation parameter ε.)

The eigenvalues are 2 and $2+\varepsilon$. The null space of $B-2I = \begin{bmatrix} 0 & 1 \\ 0 & \varepsilon \end{bmatrix}$ is the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (and its multiples). The null space of $B-(2+\varepsilon)I = \begin{bmatrix} -\varepsilon & 1 \\ 0 & 0 \end{bmatrix}$ is the vector $\begin{bmatrix} 1 \\ \varepsilon \end{bmatrix}$ (and its multiples). Thus the eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ \varepsilon \end{bmatrix}$.

c. Describe the effect of the perturbation ε on eigenvalues and eigenvectors of *A.* Comment on the linear independence of the eigenvectors of *B.*

The perturbation has introduced a second eigenvector but it is nearly linearly dependent upon the first.

d. Let $C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. What are its eigenvalues and eigenvectors? (Choose linearly independent eigenvectors.)

The eigenvalues are 2 and 2. The null space of $C-2I=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ is all of $\mathbb{R}^2$. Two linearly independent eigenvectors are $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$.

e. Let $D=\begin{bmatrix}2 & \varepsilon\\ 0 & 2\end{bmatrix}$. What are its eigenvalues and eigenvectors?

The eigenvalues are 2 and 2. The null space of $D-2I=\begin{bmatrix}0 & \varepsilon\\ 0 & 0\end{bmatrix}$ is the vector $\begin{bmatrix}1\\0\end{bmatrix}$ (and its multiples). The only eigenvector is $\begin{bmatrix}1\\0\end{bmatrix}$.

f. Describe the effect of the perturbation ε on eigenvalues and eigenvectors of C.

The perturbation has left the eigenvalues unperturbed but has removed the second eigenvector.

4. Using the diagonal form to compute high powers:

Let $A=\begin{bmatrix}1 & -2\\ -2 & 1\end{bmatrix}$. *Feel free to express answers in parts* **c**, **d**, *and* **e** *using expressions involving powers.*

a. What are its eigenvalues and eigenvectors? (SeeProblem 1, if necessary.)

The characteristic polynomial is $(1-\lambda)(1-\lambda)-4=\lambda^2-2\lambda-3=(\lambda-3)(\lambda+1)$ so the eigenvalues are 3 and -1. The null space of $A-3I=\begin{bmatrix}-2 & -2\\ -2 & -2\end{bmatrix}$ is the vector $\begin{bmatrix}1\\-1\end{bmatrix}$ (and its multiples). The null space of $A-(-1)I=\begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix}$ is the vector $\begin{bmatrix}1\\1\end{bmatrix}$ (and its multiples). Thus the eigenvectors are $\begin{bmatrix}1\\-1\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$.

b. Using part **a.,** form D, a diagonal matrix of eigenvalues, form V whose columns are the associated eigenvectors, then compute V^{-1}, and finally VDV^{-1}. Compare VDV^{-1} to A.

Since $\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}^{-1}=\begin{bmatrix}1/2 & -1/2\\ 1/2 & 1/2\end{bmatrix}$, $A=\begin{bmatrix}1 & -2\\ -2 & 1\end{bmatrix}=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1/2 & -1/2\\ 1/2 & 1/2\end{bmatrix}$.

c. Using part **b.**, what is $A^{100}y$, for $y=\begin{bmatrix}1\\2\end{bmatrix}$? (Do **not** compute A^{100} - yet. Use associativity in a clever way.)

$$A^{100}y = VD^{100}V^{-1}y = \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}3^{100} & 0\\ 0 & (-1)^{100}\end{bmatrix}\begin{bmatrix}1/2 & -1/2\\ 1/2 & 1/2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}$$
$$= \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}3^{100} & 0\\ 0 & (-1)^{100}\end{bmatrix}\begin{bmatrix}-1/2\\ 3/2\end{bmatrix}$$
$$= \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}-3^{100}/2\\ 3/2\end{bmatrix}$$
$$= \begin{bmatrix}(3-3^{100})/2\\ (3^{100}+3)/2\end{bmatrix}$$

d. Express your answer in part **c** as $A^{100}y=\gamma\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$, where γ is such that the largest component of $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ is +1. Compare $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ to the eigenvector corresponding to λ_1.

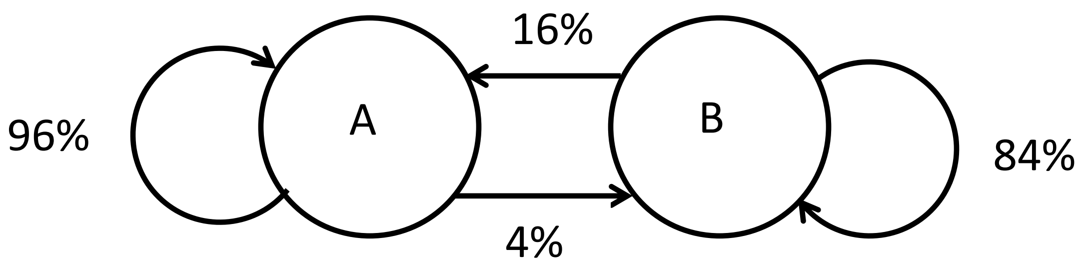
$$A^{100}y = \begin{bmatrix}(3-3^{100})/2\\ (3^{100}+3)/2\end{bmatrix} = \frac{(3^{100}+3)}{2}\begin{bmatrix}\dfrac{-3^{100}+3}{3^{100}+3}\\ 1\end{bmatrix}.$$ The vector $\begin{bmatrix}\dfrac{-3^{100}+3}{3^{100}+3}\\ 1\end{bmatrix}$ is very close (within 10^{-46}) to the negative of the eigenvector $\begin{bmatrix}1\\ -1\end{bmatrix}$.

e. Using part **b.**, what is A^{100}?

$$A^{100} = VD^{100}V^{-1} = \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}3^{100} & 0\\ 0 & (-1)^{100}\end{bmatrix}\begin{bmatrix}1/2 & -1/2\\ 1/2 & 1/2\end{bmatrix}$$
$$= \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}3^{100}/2 & -3^{100}/2\\ 1/2 & 1/2\end{bmatrix}$$
$$= \begin{bmatrix}(3^{100}+1)/2 & (-3^{100}+1)/2\\ (-3^{100}+1)/2 & (3^{100}+1)/2\end{bmatrix}.$$

5. A Markov process:

Repeat all five parts of Problem **4** with $A = \begin{bmatrix} 24/25 & 4/25 \\ 1/25 & 21/25 \end{bmatrix}$ except in part **c.** use $y = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$.



a. What are its eigenvalues and eigenvectors?

The characte3ristic polynomial is $(24/25-\lambda)(21/25-\lambda)-4/25^2 = \lambda^2 - 9/5\lambda + 4/5 = (\lambda-1)(\lambda-4/5)$ so the eigenvalues are 1 and 4/5. The null space of $A - I = \begin{bmatrix} -1/25 & 4/25 \\ 1/25 & -4/25 \end{bmatrix}$. is the vector $\begin{bmatrix} 1 \\ 1/4 \end{bmatrix}$ (and its multiples). The null space of $A - 4/5I = \begin{bmatrix} 4/25 & 4/25 \\ 1/25 & 1/25 \end{bmatrix}$ is the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (and its multiples). Thus the eigenvectors are $\begin{bmatrix} 1 \\ 1/4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

b. Using part a. express $A = VDV^{-1}$ (where the columns of V are the eigenvectors and D is a diagonal matrix containing the associated eigenvalues.)

Since $\begin{bmatrix} 1 & 1 \\ 1/4 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 4/5 & 4/5 \\ 1/5 & -4/5 \end{bmatrix}$,

$$A = \begin{bmatrix} 24/25 & 4/25 \\ 1/25 & 21/25 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1/4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4/5 \end{bmatrix} \begin{bmatrix} 4/5 & 4/5 \\ 1/5 & -4/5 \end{bmatrix}.$$

c. Using part **b.**, what is $A^{100}y$, for $y = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$? (Do **not** compute A^{100} - yet.)

$$A^{100}y = VD^{100}V^{-1}y = \begin{bmatrix} 1 & 1 \\ 1/4 & -1 \end{bmatrix}\begin{bmatrix} 1^{100} & 0 \\ 0 & (4/5)^{100} \end{bmatrix}\begin{bmatrix} 4/5 & 4/5 \\ 1/5 & -4/5 \end{bmatrix}\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1/4 & -1 \end{bmatrix}\begin{bmatrix} 1^{100} & 0 \\ 0 & (4/5)^{100} \end{bmatrix}\begin{bmatrix} 4/5 \\ -3/10 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1/4 & -1 \end{bmatrix}\begin{bmatrix} 4/5 \\ -3\cdot(4/5)^{100}/10 \end{bmatrix}$$

$$= \begin{bmatrix} 4/5 - 3\cdot(4/5)^{100}/10 \\ 1/5 + 3\cdot(4/5)^{100}/10 \end{bmatrix}$$

d. Express your answer in part **c** as $A^{100}y = \gamma\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, where γ is such that the largest component of $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is +1. Compare $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ to the eigenvector corresponding to λ_1.

$$A^{100}y == \begin{bmatrix} 4/5 - 3\cdot(4/5)^{100}/10 \\ 1/5 + 3\cdot(4/5)^{100}/10 \end{bmatrix} = (4/5 - 3\cdot(4/5)^{100}/10)\begin{bmatrix} 1 \\ \dfrac{1+15\cdot(4/5)^{100}/10}{4-15\cdot(4/5)^{100}/10} \end{bmatrix}.$$ The vector $\begin{bmatrix} 1 \\ \dfrac{1+15\cdot(4/5)^{100}/10}{4-15\cdot(4/5)^{100}/10} \end{bmatrix}$ is very close (within 10^{-10}) to the eigenvector $\begin{bmatrix} 1 \\ 1/4 \end{bmatrix}$.

e. Using part **b.**, what is A^{100}?

$$A^{100} = VD^{100}V^{-1} = \begin{bmatrix} 1 & 1 \\ 1/4 & -1 \end{bmatrix}\begin{bmatrix} 1^{100} & 0 \\ 0 & (4/5)^{100} \end{bmatrix}\begin{bmatrix} 4/5 & 4/5 \\ 1/5 & -4/5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1/4 & -1 \end{bmatrix}\begin{bmatrix} 4/5 & 4/5 \\ (4/5)^{100}/5 & -(4/5)^{101} \end{bmatrix}$$

$$= \begin{bmatrix} 4/5 + (4/5)^{100}/5 & 4/5 - (4/5)^{101} \\ 1/5 - (4/5)^{100}/5 & 1/5 + (4/5)^{101} \end{bmatrix}.$$

6. All zero eigenvalues:

Find a simple non-zero matrix having all zero eigenvalues.

The matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a non-zero matrix having all zero eigenvalues