Examination 2 Solutions

CS 313H

1. [10] Using induction, prove that for $n \geq 0, \sum_{k=0}^{n} 2^k = 2^{n+1} - 1$

For $n = 1,$ we have $\sum_{k=0}^{0} 2^k = 2^0 = 1 = 2^{0+1} - 1.$ Now assume the result is true for $n \geq 0.$ We then have $\sum_{k=0}^{n+1} 2^k = \sum_{k=0}^{n} 2^k + 2^{n+1} = (2^{n+1} - 1) + 2^{n+1} = 2 \cdot 2^{n+1} - 1 = 2^{(n+1)+1} - 1.$ The result then holds for all $n \geq 0.$

2.a [5] Prove for any set $A,$ that $A = A \cap A$.

We have

$$
\begin{aligned}
& x \in A \\
& \Leftrightarrow x \in A \wedge x \in A \\
& \Leftrightarrow x \in A \cap A.
\end{aligned}
$$

Thus, $A = A \cap A$

b [5] Prove for any set $A,$ that $A = A \cup A$.

We have

$$
\begin{aligned}
& x \in A \\
& \Leftrightarrow x \in A \vee x \in A \\
& \Leftrightarrow x \in A \cup A.
\end{aligned}
$$

Thus, $A = A \cup A$

b [10] Using parts a. and b. to prove that $A \cup B = A \cap B$ if and only if $A = B.$

Suppose $A = B.$ Then $A \cup B = A \cup A = A = A \cap A = A \cap B.$ Alternatively, assume $A \cup B = A \cap B$. Then $A \subseteq A \cup B = A \cap B \subseteq B$ and $B \subseteq A \cup B = A \cap B \subseteq A$, so $A = B.$

3. [15]. Given sets $A, B,$ and C and relations R, between A and B, and S, between B and C, prove that $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

We have

$$
\begin{aligned}
&(x, y) \in (S \circ R)^{-1} \\
&\Leftrightarrow (y, x) \in S \circ R \\
&\Leftrightarrow \exists z \in B \ni (y, z) \in R \wedge (z, x) \in S \\
&\Leftrightarrow \exists z \in B \ni (x, z) \in S^{-1} \wedge (z, y) \in R^{-1} \\
&\Leftrightarrow (y, x) \in R^{-1} \circ S^{-1}.
\end{aligned}
$$

Thus, $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

4. [20]. Prove that if R is a partial order on a set A, then R^{-1} is also a partial order on A.

We need to show that R^{-1} is reflexive, antisymmetric, and transitive. Since R is a partial order, thus R is reflexive, antisymmetric, and transitive, we have first that for all $x \in A, (x, x) \in R,$ so $(x, x) \in R^{-1}$, and R^{-1} is reflexive. Now take $x, y \in A$ so that $(x, y) \in R^{-1}$ and $(y, x) \in R^{-1}$. We then have $(y, x) \in R$ and $(x, y) \in R$ so by the antisymmetry of R, $x \neq y$. Thus, R^{-1} is antisymmetric. Finally, take $x, y, z \in A$ so that $(x, y) \in R^{-1}$ and $(y, z) \in R^{-1}$. We then have $(y, x) \in R$ and $(z, y) \in R$ so by the transitivity of R, $(z, x) \in R$ and hence $(x, z) \in R^{-1}$. Thus, R^{-1} is transitive and a partial order.

5.. Consider relations R and S on a set A . Prove or disprove with a simple counter example each of the following:

.a [10] If R and S are symmetric, then $R \sim S$ is symmetric.

If $(x, y) \in R \sim S$, then $(x, y) \in R$ and by the symmetry of R, $(y, x) \in R$. Also, if $(x, y) \in R \sim S$ then $(x, y) \notin S$ and by the symmetry of R, $(y, x) \notin S$. We conclude that $(y, x) \in R \sim S$ so $R \sim S$ is symmetric.

b [10] If R and S are antisymmetric, then $R \sim S$ is antisymmetric.

If $(x, y) \in R \sim S$ and $(y, x) \in R \sim S$, then $(x, y) \in R$ and $(y, x) \in R$ so by the antisymmetry of R, $x \neq y$.. We conclude that $R \sim S$ is antisymmetric.

b [10] If R and S are transitive, then $R \sim S$ is transitive.

$R \sim S$ may not be transitive. Let $A = \{1, 2\}, R = \{(1,1), (1,2), (2,1), (2,2)\}$, and $S = \{(1,1), (2,2)\}$. Then $R \sim S = \{(1,2), (2,1)\}$ and $(1,2) \in R \sim S$ and $(1,2) \in R \sim S$, yet $(1,1) \notin R \sim S$ so $R \sim S$ is not transitive.