Practice Examination 2 Solutions

CS 313H

1. [15] Prove that $((\exists x)Px \wedge A)$ follows from $(\exists x)(Px \wedge A)$

$\{P_1\}$	(1). $(\exists x)(Px \wedge A)$	P
$\{P_1\}$	(2). $Pa \wedge A$	EI (1)
$\{P_1\}$	(3). Pa	Simp (2)
$\{P_1\}$	(4). $(\exists x)Px$	EG (3)
$\{P_1\}$	(5). A	Simp (2)
$\{P_1\}$	(6). $((\exists x)Px \wedge A)$	Add (1), (5)

2. [10] For any sets A and B, prove that $A \sim (A \sim B) = A \cap B$.

$x \in A \sim (A \sim B)$
$\Rightarrow x \in A \wedge \sim (x \in A \sim B)$
$\Rightarrow x \in A \wedge \sim (x \in A \wedge x \notin B)$
$\Rightarrow x \in A \wedge (x \notin A \vee x \in B)$
$\Rightarrow (x \in A \wedge x \notin A) \vee (x \in A \wedge x \in B)$
$\Rightarrow x \in A \wedge x \in B$
$\Rightarrow x \in A \cap B.$

$x \in A \cap B$
$\Rightarrow x \in A \wedge x \in B$
$\Rightarrow x \in A \wedge x \in B$
$\Rightarrow x \in A \wedge (x \notin A \vee x \in B)$
$\Rightarrow x \in A \wedge \sim (x \in A \wedge x \notin B)$
$\Rightarrow x \in A \wedge \sim (x \in A \sim B)$
$\Rightarrow x \in A \sim (A \sim B).$

3. [20] Using induction prove for $n \geq 2$, that $\prod_{k=2}^{n}(1-\frac{1}{k^2}) = \frac{n+1}{2n}$.

For $n = 2$, we have

$$\begin{aligned}
\prod_{k=2}^{n}(1-\frac{1}{k^2}) &= \prod_{k=2}^{2}(1-\frac{1}{k^2}) \\
&= 1-\frac{1}{4} \\
&= \frac{3}{4} \\
&= \frac{2+1}{2\cdot 2} \\
&= \frac{n+1}{2n}.
\end{aligned}$$

Now assume for some $n \geq 2$, that $\prod_{k=2}^{n}(1-\frac{1}{k^2}) = \frac{n+1}{2n}$. We then have

$$\begin{aligned}
\prod_{k=2}^{n+1}(1-\frac{1}{k^2}) &= \prod_{k=2}^{n}(1-\frac{1}{k^2})(1-\frac{1}{(n+1)^2}) \\
&= \frac{n+1}{2n}(1-\frac{1}{(n+1)^2}) \\
&= \frac{n+1}{2n}\frac{(n+1)^2-1}{(n+1)^2} \\
&= \frac{n+1}{2n}\frac{n(n+2)}{(n+1)^2} \\
&= \frac{n+2}{2(n+1)} \\
&= \frac{(n+1)+1}{2(n+1)}.
\end{aligned}$$

4. [10] For any sets A and B, prove that $P(A \cap B) = P(A) \cap P(B)$.

$$\begin{aligned}
& X \in P(A \cap B) \\
\Leftrightarrow\ & X \subseteq A \cap B \\
\Leftrightarrow\ & X \subseteq A \wedge X \subseteq B \\
\Leftrightarrow\ & X \in P(A) \wedge X \in P(B) \\
\Leftrightarrow\ & X \in P(A) \cap P(B).
\end{aligned}$$

5. [10] Given a set A and two symmetric relations R and S on A, prove or disprove with a simple counter-example: $R \circ S$ is symmetric.

> $R \circ S$ need not be symmetric. Let $A = \{1,2,3\}$, $S = \{(1,2),(2,1)\}$, and $R = \{(2,3),(3,2)\}$. Then $R \circ S = \{(1,3)\}$ so $(1,3) \in R \circ S$ but $(3,1) \in R \circ S$.

6. [20] Consider the relation R on $\mathbb{Z}$, the set of integers: $R = \{(x, y) : x + y \text{ iseven}\}$. Prove that R is an equivalence relation.

> We seek to show R is reflexive, symmetric, and transitive. To that end, consider any $x \in \mathbb{Z}$. Since $x + x = 2x$ is even, $(x, x) \in R$ and R is reflexive. If $(x, y) \in R$ then $x + y$ is even, thus $y + x$ is even, $(y, x) \in R$, and R is symmetric. Lastly, suppose $(x, y) \in R$ and $(y, z) \in R$ thus both $x + y$ and $y + z$ are even. The sum $x + z + 2y$ is even as well as $2y$, so the difference $x + z = (x + z + 2y) - 2y$ is even so $(x, z) \in R$ and R is transitive. We conclude that R is an equivalence relation.