Examination 2 Solutions

1. [15] Using induction, prove that for \( n \geq 1 \), \( n^3 + 2n \) is an integral multiple of 3 (i.e. \( \forall n \in \mathbb{Z}((n \geq 1) \Rightarrow \exists k \in \mathbb{Z}(n^3 + 2n = 3k)) \).

   For \( n = 1 \), we have \( n^3 + 2n = 3 \cdot 1 \). Now assume the result is true for \( n \geq 1 \). We then have some integer \( k \) so that \( n^3 + 2n = 3k \). But then
   \[
   (n+1)^3 + 2(n+1) = n^3 + 3n^2 + 3n + 1 + 2n + 2
   \]
   \[
   = n^3 + 2n + 3(n^2 + n + 1)
   \]
   \[
   = 3k + 3(n^2 + n + 1)
   \]
   \[
   = 3 \cdot (k + n^2 + n + 1).
   \]
   Since \( n \) is an integer, so is \( k + n^2 + n + 1 \) so \( (n+1)^3 + 2(n+1) \) is an integral multiple of 3. The result then holds for all \( n \geq 1 \).

2. [10] Using induction, prove that for \( n \geq 1 \), \( \sum_{k=1}^{n} (4k - 3) = n(2n - 1) \).

   For \( n = 1 \), we have \( \sum_{k=1}^{n} (4k - 3) = 4 - 3 = 1 = 1(2 \cdot 1 - 1) = n(2n - 1) \). Now assume the result is true for \( n \geq 1 \). We then have
   \[
   \sum_{k=1}^{n+1} (4k - 3) = \sum_{k=1}^{n} (4k - 3) + (4(n+1) - 3)
   \]
   \[
   = n(2n-1) + 4n + 4 - 3
   \]
   \[
   = 2n^2 - n + 4n + 1
   \]
   \[
   = 2(n+1)^2 - (n+1)
   \]
   \[
   = (n+1)(2(n+1) - 1).
   \]
   The result then holds for all \( n \geq 1 \).

3. a[5] Prove for any sets \( A, B, \) and \( C \), that \( (A \cup B) \sim C = (A \sim C) \cup (B \sim C) \).

   \( x \in (A \cup B) \sim C \)
   \[\iff (x \in A) \lor (x \in B) \land x \notin C \]
   \[\iff (x \in A) \land x \notin C \lor (x \in B) \land x \notin C \]
   \[\iff x \in A \sim C \lor x \in B \sim C \]
   \[\iff x \in (A \sim C) \cup (B \sim C) \]

   b.[10] Using induction and part a, prove for \( n \geq 1 \), all sets \( A_1, A_2, \ldots, A_n \), and all \( C \):
   \[
   \left( \bigcup_{i=1}^{n} A_i \right) \sim C = \bigcup_{i=1}^{n} (A_i \sim C) .
   \]
For $n = 1$, we have $(\bigcup_{i=1}^{n} A_i) \sim C = A_1 \sim C = \bigcup_{i=1}^{n} (A_i \sim C)$. Now assume the result is true for $n \geq 1$. We then have

$$(\bigcup_{i=1}^{n+1} A_i) \sim C = (\bigcup_{i=1}^{n} A_i \cup A_{n+1}) \sim C$$

$$= ((\bigcup_{i=1}^{n} A_i) \sim C) \cup (A_{n+1} \sim C)$$

$$= \bigcup_{i=1}^{n} (A_i \sim C) \cup (A_{n+1} \sim C)$$

$$= \bigcup_{i=1}^{n+1} (A_i \sim C).$$

The result then holds for all $n \geq 1$. 
4. [15] Using induction, prove a generalized distributivity law for sets – that is, for \( n \geq 1 \) and all sets \( A \) and \( B_1, B_2, \ldots, B_n \),

\[
A \cup \bigcap_{i=1}^n B_i = \bigcap_{i=1}^n (A \cup B_i).
\]

(Recall that \( \bigcap_{i=1}^{n+1} B_i = (\bigcap_{i=1}^n B_i) \cap B_{n+1} \).)

For \( n = 1 \), we have \( A \cup \bigcap_{i=1}^1 B_i = A \cup B_1 \bigcap_{i=1}^1 (A \cup B_i) \). Now assume the result is true for \( n \geq 1 \). We then have

\[
A \cup \bigcap_{i=1}^{n+1} B_i = A \cup (\bigcap_{i=1}^n B_i \cap B_{n+1})
\]

\[
= (A \cup \bigcap_{i=1}^n B_i) \cap (A \cup B_{n+1})
\]

\[
= \bigcap_{i=1}^n (A \cup B_i) \cap (A \cup B_{n+1})
\]

\[
= \bigcap_{i=1}^{n+1} (A \cup B_i)
\]

and the result then holds for all \( n \geq 1 \).

5. [10] For all sets \( A, B, C, \) and \( D \), prove that \((A \cap B) \times (C \cap D) \subseteq (A \times C) \cap (B \times D)\).

We have
\[
(x, y) \in (A \cap B) \times (C \cap D)
\]
\[
\Rightarrow x \in A \cap B \land y \in C \cap D
\]
\[
\Rightarrow x \in A \land x \in B \land y \in C \land y \in D
\]
\[
\Rightarrow x \in A \land y \in C \land x \in B \land y \in D
\]
\[
\Rightarrow (x, y) \in A \times C \land (x, y) \in B \times D
\]
\[
\Rightarrow (x, y) \in (A \times C) \cap (B \times D)
\]

6. a [10]. Either prove or give a simple counterexample. Given symmetric relations \( R \) and \( S \) on a set \( A \), the composition \( S \circ R \) is symmetric. (If you present a counterexample, present the relations as specific sets of ordered pairs rather than using matrices or graphs.)

This is false. Let \( A = \{0, 1\}, R = \{(0, 1), (1, 0)\}, \) and \( S = \{(0, 0)\} \). We have \( S \circ R = \{(1, 0)\} \).

Since \((1, 0) \in S \circ R \) but \((0, 1) \notin S \circ R \), \( S \circ R \) is not symmetric.

b [10]. Either prove or give a simple counterexample. Given antisymmetric relations \( R \) and \( S \) on a set \( A \), the difference \( R \sim S \) is antisymmetric. (If you present a counterexample, present the relations as specific sets of ordered pairs rather than using matrices or graphs.)
This is true. We have
\((x, y) \in R \sim S \land x \neq y\)
\(\Rightarrow (x, y) \in R \land x \neq y\)
\(\Rightarrow (y, x) \not\in R\)
\(\Rightarrow (y, x) \not\in R \sim S.\)