Examination 2 Solutions

1. [15] Using induction, prove that for $n \geq 1$, $n^3 + 2n$ is an integral multiple of 3 (i.e. $\forall n \in \mathbb{Z}((n \geq 1) \Rightarrow \exists k \in \mathbb{Z}(n^3 + 2n = 3k))$.

For $n = 1$, we have $n^3 + 2n = 3 = 3 \cdot 1$. Now assume the result is true for $n \geq 1$. We then have some integer k so that have $n^3 + 2n = 3k$. But then

$$\begin{aligned}(n+1)^3 + 2(n+1) &= n^3 + 3n^2 + 3n + 1 + 2n + 2 \\ &= n^3 + 2n + 3(n^2 + n + 1) \\ &= 3k + 3(n^2 + n + 1) \\ &= 3 \cdot (k + n^2 + n + 1).\end{aligned}$$

Since n is an integer, so is $k + n^2 + n + 1$ so $(n+1)^3 + 2(n+1)$ is an integral multiple of 3. The result then holds for all $n \geq 1$.

2. [10] Using induction, prove that for $n \geq 1$, $\sum_{k=1}^{n} (4k - 3) = n(2n-1)$.

For $n = 1$, we have $\sum_{k=1}^{n} (4k-3) = 4 - 3 = 1 = 1(2 \cdot 1 - 1) = n(2n-1)$. Now assume the result is true for $n \geq 1$. We then have

$$\begin{aligned}\sum_{k=1}^{n+1} (4k-3) &= \sum_{k=1}^{n} (4k-3) + (4(n+1) - 3) \\ &= n(2n-1) + 4n + 4 - 3 \\ &= 2n^2 - n + 4n + 1 \\ &= 2(n+1)^2 - (n+1) \\ &= (n+1)(2(n+1) - 1).\end{aligned}$$

The result then holds for all $n \geq 1$.

3. a[5]Prove for any sets A, B, and C, that $(A \cup B) \sim C = (A \sim C) \cup (B \sim C)$.

$$\begin{aligned}&x \in (A \cup B) \sim C \\ \Leftrightarrow\ &(x \in A \vee x \in B) \wedge x \notin C \\ \Leftrightarrow\ &(x \in A \wedge x \notin C) \vee (x \in B \wedge x \notin C) \\ \Leftrightarrow\ &x \in A \sim C \vee x \in B \sim C \\ \Leftrightarrow\ &x \in (A \sim C) \cup (B \sim C)\end{aligned}$$

b.[10]Using induction and part a, prove for $n \geq 1$, all sets $A_1, A_2, \ldots, A_n$, and all C:

$$(\bigcup_{i=1}^{n} A_i) \sim C = \bigcup_{i=1}^{n} (A_i \sim C).$$

For $n=1$, we have $(\bigcup_{i=1}^{n} A_i) \sim C = A_1 \sim C = \bigcup_{i=1}^{n} (A_i \sim C)$. Now assume the result is true for $n \geq 1$. We then have

$$\begin{aligned}
(\bigcup_{i=1}^{n+1} A_i) \sim C &= (\bigcup_{i=1}^{n} A_i \cup A_{n+1}) \sim C \\
&= ((\bigcup_{i=1}^{n} A_i) \sim C) \cup (A_{n+1} \sim C) \\
&= \bigcup_{i=1}^{n} (A_i \sim C) \cup (A_{n+1} \sim C) \\
&= \bigcup_{i=1}^{n+1} (A_i \sim C).
\end{aligned}$$

The result then holds for all $n \geq 1$.

4. [15] Using induction, prove a generalized distributivity law for sets – that is, for $n \geq 1$ and all sets A and $B_1, B_2, ..., B_n$,

$$A \cup \bigcap_{i=1}^{n} B_i = \bigcap_{i=1}^{n} (A \cup B_i).$$

(Recall that $\bigcap_{i=1}^{n+1} B_i = (\bigcap_{i=1}^{n} B_i) \cap B_{n+1}.$)

For $n = 1,$ we have $A \cup \bigcap_{i=1}^{1} B_i = A \cup B_1 = \bigcap_{i=1}^{1} (A \cup B_i).$ Now assume the result is true for $n \geq 1.$ We then have

$$\begin{aligned} A \cup \bigcap_{i=1}^{n+1} B_i &= A \cup (\bigcap_{i=1}^{n} B_i \cap B_{n+1}) \\ &= (A \cup \bigcap_{i=1}^{n} B_i) \cap (A \cup B_{n+1}) \\ &= \bigcap_{i=1}^{n} (A \cup B_i) \cap (A \cup B_{n+1}) \\ &= \bigcap_{i=1}^{n+1} (A \cup B_i) \end{aligned}$$

and the result then holds for all $n \geq 1.$

5. [10] For all sets $A, B, C,$ and D, prove that $(A \cap B) \times (C \cap D) \subseteq (A \times C) \cap (B \times D)$.

We have
$(x, y) \in (A \cap B) \times (C \cap D)$
$\Rightarrow x \in A \cap B \wedge y \in C \cap D$
$\Rightarrow x \in A \wedge x \in B \wedge y \in C \wedge y \in D$
$\Rightarrow x \in A \wedge y \in C \wedge x \in B \wedge y \in D$
$\Rightarrow (x, y) \in A \times C \wedge (x, y) \in B \times D.$
$\Rightarrow (x, y) \in (A \times C) \cap (B \times D)$

6. a [10]. Either prove or give a simple counterexample. Given symmetric relations R and S on a set A, the composition $S \circ R$ is symmetric. (If you present a counterexample, present the relations as specific sets of ordered pairs rather using matrices or graphs.)

This is false. Let $A = \{0,1\}, R = \{(0,1),(1,0)\}$, and $S = \{(0,0)\}$. We have $S \circ R = \{(1,0)\}$. Since $(1,0) \in S \circ R$ but $(0,1) \notin S \circ R$, $S \circ R$ is not symmetric.

b [10]. Either prove or give a simple counterexample. Given antisymmetric relations R and S on a set A, the difference $R \sim S$ is antisymmetric. (If you present a counterexample, present the relations as specific sets of ordered pairs rather using matrices or graphs.)

This is true. We have

$$(x, y) \in R \sim S \wedge x \neq y$$

$$\Rightarrow (x, y) \in R \wedge x \neq y$$

$$\Rightarrow (y, x) \notin R$$

$$\Rightarrow (y, x) \notin R \sim S.$$