Examination 2 Solutions

1. [10] For fixed real numbers a and b, consider the iteratively defined sequence:

$$s_0 = a$$
$$s_n = 2s_{n-1} + b, \; for \; n \geq 1.$$

Using induction, prove that for $n \geq 0, s_n = 2^n a + (2^n - 1)b$.

For $n = 0$, we have $s_0 = a = 1 \cdot a + (1-1)b = 2^0 a + (2^0 - 1)b$. Now assume the result is true for some $n \geq 0$. We then have

$$\begin{aligned} s_{n+1} &= 2s_n + b \\ &= 2(2^n a + (2^n - 1)b) + b \\ &= 2^{n+1} a + (2^{n+1} - 2 + 1)b \\ &= 2^{n+1} a + (2^{n+1} - 1)b. \end{aligned}$$

The result then holds for $n+1$ and by induction holds for all $n \geq 0$.

2. [10] Using induction, prove that for $n \geq 4$, $n! > 2^n$.

For $n = 4$, we have $n! = 4! = 24 > 16 = 2^4 = 2^n$. Now assume the result is true for some $n \geq 4$. We then have $n + 1 \geq 5 > 2$ so

$$(n+1)! = (n+1) \cdot n! > (n+1) \cdot 2^n > 2 \cdot 2^n = 2^{n+1}.$$

The result then holds for $n+1$ and by induction holds for all $n \geq 4$.

3. [10] Prove for any sets $A, B,$ and C, that if $A \subseteq C \cup B$ then $A \sim C \subseteq B$.

$$\begin{aligned} & x \in A \sim C \\ \Rightarrow\; & x \in A \wedge x \notin C \\ \Rightarrow\; & x \in C \cup B \wedge x \notin C \\ \Rightarrow\; & (x \in C \vee x \in B) \wedge x \notin C \\ \Rightarrow\; & (x \in C \wedge x \notin C) \vee (x \in B \wedge x \notin C) \\ \Rightarrow\; & x \in B \wedge x \notin C \\ \Rightarrow\; & x \in B. \end{aligned}$$

4. [10]. Given sets $A, B,$ and C, prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

$$(x, y) \in A \times (B \cup C)$$
$$\Leftrightarrow x \in A \wedge y \in B \cup C$$
$$\Leftrightarrow x \in A \wedge (y \in B \vee y \in C)$$
$$\Leftrightarrow (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C)$$
$$\Leftrightarrow (x, y) \in (A \times B) \vee (x, y) \in (A \times C)$$
$$\Leftrightarrow (x, y) \in (A \times B) \cup (A \times C).$$

5. Show these two definitions of antisymmetry for a relation R on a set A are equivalent:

a. $\forall x, y \in A : ((x, y) \in R \wedge x \neq y) \Rightarrow (y, x) \notin R.$

b. $\forall x, y \in A : ((x, y) \in R \wedge (y, x) \in R) \Rightarrow x = y.$

(Hint: Ignore the universal quantifier, let $P = "(x, y) \in R", Q = "(y, x) \in R",$ and $E = "x = y"$. Use simple logical identities to convert one to the other.)

Using the suggested notation, definition **a** is $(P \wedge \sim E) \Rightarrow \sim Q$ and definition **b** is $(P \wedge Q) \Rightarrow E$. We have:

$$(P \wedge \sim E) \Rightarrow \sim Q$$
$$\Leftrightarrow \sim (P \wedge \sim E) \vee \sim Q$$
$$\Leftrightarrow (\sim P \wedge \sim\sim E) \vee \sim Q$$
$$\Leftrightarrow \sim P \vee E \vee \sim Q$$
$$\Leftrightarrow \sim P \vee \sim Q \vee E.$$
$$\Leftrightarrow \sim (P \wedge Q) \vee E$$
$$\Leftrightarrow (P \wedge Q) \Rightarrow E$$

6. For these problems either prove the claim or give a simple counterexample. If you present a counterexample, present the relations as specific sets of ordered pairs rather than using matrices or graphs. For assume R and S are relations on a set A and $R \subseteq S$.

a [10]. If R is transitive then S is transitive.

This is false. Let $A = \{0,1\}, R = \varnothing,$ and $S = \{(0,1),(1,0)\}$. R is transitive and $R \subseteq S$. Since $(0,1) \in S$ and $(1,0) \in S$ but $(0,0) \notin S$, S is not transitive.

b [10]. If S is antisymmetric then R is antisymmetric. (Note the reversal of the order from part a.)

This is true. We have by the antisymmetry of S,
$(x, y) \in R \wedge x \neq y$
$\Rightarrow (x, y) \in S \wedge x \neq y$
$\Rightarrow (y, x) \notin S$
$\Rightarrow (y, x) \notin R.$
So R is also antisymmetric.

c [10]. If R is transitive then $R \circ R$ is transitive.

This is true. We have by the transitivity of R,
$(x, y) \in R \circ R \wedge (y, z) \in R \circ R$
$\Rightarrow \exists u, v \in A \ni (x, u) \in R \wedge (u, y) \in R \wedge (y, v) \in R \wedge (v, z) \in R$
$\Rightarrow (x, y) \in R \wedge (y, z) \in R$
$\Rightarrow (x, z) \in R \circ R.$
So $R \circ R$ is also transitive.

d [10]. If R is symmetric then $R \circ R$ is symmetric.

This is true. We have by the symmetry of R,
$(x, y) \in R \circ R$
$\Rightarrow \exists z \in A \ni (x, z) \in R \wedge (z, y) \in R$
$\Rightarrow \exists z \in A \ni (z, x) \in R \wedge (y, z) \in R$
$\Rightarrow (y, x) \in R \circ R.$
So $R \circ R$ is also symmetric.