CS 313H

Examination 2 Solutions

1. [10] For fixed real numbers a and b, with $a \neq 1$, define

$$x_0 = 0,$$

and for $k = 1, 2, \ldots$

$$x_k = a\,x_{k-1} + b.$$

Using induction, prove that for $k \geq 0$, $x_k = \dfrac{a^k - 1}{a-1} b.$

Consider the inductive hypothesis, $P(k) = "x_k = \dfrac{a^k - 1}{a-1} b"$. For $k = 0$, we have

$x_0 = 0 = \dfrac{a^0 - 1}{a-1} b.$ So $P(0)$ is true. Now for some $k \geq 1$, assume $P(k)$. We then have

$$\begin{aligned} x_{k+1} &= a\,x_k + b \\ &= a\frac{a^k - 1}{a-1} b + b \\ &= \frac{a^{k+1} - a + a - 1}{a-1} b \\ &= \frac{a^{k+1} - 1}{a-1} b, \end{aligned}$$

so $P(k+1)$ is also true. By induction we have for $k \geq 0$, $x_k = \dfrac{a^k - 1}{a-1} b.$

2. [10] Prove for any sets $A, B,$ and $C,$ that if $A \subseteq B$ then $C \sim B \subseteq C \sim A$.

Assuming $A \subseteq B$, we have

$$\begin{aligned} & x \in C \sim B \\ & \Rightarrow x \in C \wedge x \notin B \\ & \Rightarrow x \in C \wedge x \notin A \\ & \Rightarrow x \in C \sim A. \end{aligned}$$

Thus, $C \sim B \subseteq C \sim A$.

3. [15] Using induction, prove a generalized distributivity law for sets – that is, for $n \geq 1$ and all sets A and $B_1, B_2, ..., B_n$,

$$A \cap \bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} (A \cap B_i)$$

(Recall for $n \geq 1$ that $\bigcup_{i=1}^{n+1} B_i = (\bigcup_{i=1}^{n} B_i) \cup B_{n+1}.$)

Consider the inductive hypothesis, $P(n) = "A \cap \bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} (A \cap B_i)"$. $P(1)$ holds since $A \cap \bigcup_{i=1}^{1} B_i = A \cap B_1 = \bigcup_{i=1}^{1} (A \cap B_i)$. Now for some $n \geq 1$, assume $P(n)$. We then have

$$\begin{aligned} A \cap \bigcup_{i=1}^{n+1} B_i &= A \cap (\bigcup_{i=1}^{n} B_i \cup B_{n+1}) \\ &= (A \cap \bigcup_{i=1}^{n} B_i) \cup (A \cap B_{n+1}) \\ &= \bigcup_{i=1}^{n} (A \cap B_i) \cup (A \cap B_{n+1}) \\ &= \bigcup_{i=1}^{n+1} (A \cap B_i) \end{aligned}$$

so $P(n+1)$ is also true. By induction, the result then holds for all $n \geq 1$.

4. [15]. You are given a relation R on a set A. Using induction, prove that if R is transitive then for $k \geq 1$.

$$R^k \subseteq R.$$

(Recall $R^1 = R$ and for $k \geq 1$, $R^{k+1} = R \circ R^k$.)

Consider the inductive hypothesis, $P(k) = "R^k \subseteq R"$. $P(1)$ holds since $R^1 = R$ is given to be transitive. Now for some $k \geq 1$, assume $P(k)$. We then have for all $x, y \in A$,

$$\begin{aligned} &(x, y) \in R^{k+1} \\ &\Rightarrow \exists z\, (x, z) \in R^k \wedge (z, y) \in R \\ &\Rightarrow \exists z\, (x, z) \in R \wedge (z, y) \in R \\ &\Rightarrow \exists z\, (x, z) \in R. \end{aligned}$$

so $P(n+1)$ is also true. The second line follows by definition. The third line follows by the inductive hypothesis. The last line follows by transitivity. By induction, the result then holds for all $n \geq 1$.

5. **[15]**. A relation R on a set A is *countertransitive* if and only if for all $x, y, z \in A$, $((x, y) \in R \wedge (y, z) \in R) \Rightarrow (z, x) \in R$.

a. Prove that if R is symmetric and transitive then R is also countertransitive.

We have for all $x, y, z \in A$, $(x, y) \in R$ and $(y, z) \in R$ then by symmetry $(y, x) \in R$ and $(z, y) \in R$ and hence by transitivity $(z, x) \in R$. Thus, R is countertransitive.

b. Prove that if R is countertransitive and reflexive then R is also symmetric.

We have for all $x, y \in A$, if $(x, y) \in R$ then $(y, y) \in R$ by reflexivity and thus $(x, y) \in R$ $(y, x) \in R$ by countertransitivity. Thus, R is symmetric.

c. Prove that if R is countertransitive and reflexive then R is also transitive.

We have for all $x, y, z \in A$, $(x, y) \in R$ and $(y, z) \in R$ then from part b, it is symmetric, so $(y, x) \in R$ and $(z, y) \in R$. By countertransitivity $(x, y) \in R$. Thus, R is transitive.