Final Examination Solutions

CS 313H

1. [5] Use a truth table to determine for which truth values of $p, q,$ and r $(\sim p \vee q) \wedge (r \Leftrightarrow p)$ is true.

p	q	r	$\sim p$	$\sim p \vee q$	$r \Leftrightarrow p$	$(\sim p \vee q) \wedge (r \Leftrightarrow p)$
F	F	F	T	T	T	T
F	F	T	T	T	F	F
F	T	F	T	T	T	T
F	T	T	T	T	F	F
T	F	F	F	F	F	F
T	F	T	F	F	T	F
T	T	F	F	T	F	F
T	T	T	F	T	T	T

2. [15] Using sentential calculus (with a four column format), prove that the conclusion $(s \wedge \sim p) \Rightarrow t$ follows from premises: $\sim (q \wedge s)$ and $q \vee p$. (Hint: Employ Conditionalization i.e., "Rule C".)

$\{Pr_1\}$	1. $s \wedge \sim p$	P (for CP)
$\{Pr_1\}$	2 s	Simp (1)
$\{Pr_2\}$	3. $\sim (q \wedge s)$	P
$\{Pr_2\}$	4. $\sim q \vee \sim s$	DeM (3)
$\{Pr_1, Pr_2\}$	5. $\sim q$	DS (2), (4)
$\{Pr_3\}$	6. $q \vee p$	P
$\{Pr_1, Pr_2, Pr_3\}$	7. p	DS (5), (6)
$\{Pr_1\}$	8. $\sim p$	Simp (1)
$\{Pr_1, Pr_2, Pr_3\}$	9. t	ContraPrm (7), (8)
$\{Pr_2, Pr_3\}$	10. $(s \wedge \sim p) \Rightarrow t$	C (1), (9)

3. [15] Prove that the conclusion $p \Rightarrow s$ follows from the premises $\sim(p \wedge q), p \Rightarrow (q \vee r), \text{and } r \Rightarrow \sim p$. First convert the premises and the negation of the conclusion into Conjunctive Normal Form, and then employ a resolution proof to get a contradiction.

$\sim(p \wedge q)$
$\sim p \vee \sim q$

$p \Rightarrow (q \vee r)$
$\sim p \vee (q \vee r)$
$\sim p \vee q \vee r$

$r \Rightarrow \sim p$
$\sim r \vee \sim p$

$\sim(p \Rightarrow s)$
$\sim(\sim p \vee s)$
$\sim\sim p \wedge \sim s$
$p \wedge \sim s$

1. $\sim p \vee \sim q$	P
2. $\sim p \vee q \vee r$	P
3. $\sim r \vee \sim p$	P
4. p	P
5. $\sim s$	P
6. $\sim r$	Res (3), (4)
7. $\sim p \vee q$	Res (2), (6)
8. $\sim p$	Res (1), (7)
9. *false*	Conj. (4), (8)

4. [15] Using the predicates defined on the set $\mathbb{N}$ of natural numbers:

Sxy x is a the successor of y (i.e. $x = y + 1$),
Exy x equal to y

Express in the syntax of Predicate Calculus:

a. *No natural number is a successor of itself*

$(\forall x)(\sim Sxx)$

b. *Every natural number has one and only one successor.*

$(\forall y)((\exists x)(Sxy \wedge (\forall z)(Szy \Rightarrow Exz)))$

c. *b is the successor of the successor of a.*

$(\exists x)(Sxa \wedge Sbx)$

5. [20] Prove that $(\exists z)Lz$ follows from $(\forall y)(\exists x)((Lx \Rightarrow Nx) \Rightarrow Gy)$ and $(\exists x)(\sim Gx)$.

$\{P_1\}$	(1). $(\forall y)(\exists x)((Lx \Rightarrow Nx) \Rightarrow Gy)$	P
$\{P_2\}$	(2). $(\exists x)(\sim Gx)$	P
$\{P_2\}$	(3). $\sim Gb$	EI (2)
$\{P_1\}$	(4). $(\exists x)((Lx \Rightarrow Nx) \Rightarrow Gb)$	UI (1)
$\{P_1\}$	(5). $(La \Rightarrow Na) \Rightarrow Gb$	EI (1)
$\{P_1, P_2\}$	(6). $\sim (La \Rightarrow Na)$	MT (3), (5)
$\{P_1, P_2\}$	(7). $\sim (\sim La \vee Na)$	DS (6)
$\{P_1, P_2\}$	(8). $\sim\sim La \wedge \sim Na$	DM (7)
$\{P_1, P_2\}$	(9). $\sim\sim La$	Simp (8)
$\{P_1, P_2\}$	(10). La	DN (9)
$\{P_1, P_2\}$	(11). $(\exists z)Lz$	EG (10)

6a. [10] Using induction, prove that for $n \geq 0$, $\sum_{k=0}^{n}(2k+1) = (n+1)^2$.

For $n \geq 0$, let $P(n) =$ "$\sum_{k=0}^{n}(2k+1) = (n+1)^2$".

Basis step: $P(0)$ is true since $\sum_{k=0}^{0}(2k+1) = (2 \cdot 0+1) = 1 = (0+1)^2$.

Inductive step: For $n \geq 0$, $P(n) \Rightarrow P(n+1)$, since if $\sum_{k=0}^{n}(2k+1) = (n+1)^2$, then

$$\begin{aligned}\sum_{k=0}^{n+1}(2k+1) &= \sum_{k=0}^{n}(2k+1) + 2(n+1)+1 \\ &= (n+1)^2 + 2(n+1)+1 \\ &= ((n+1)+1)^2.\end{aligned}$$

7. [10] Using induction, prove that for any real number a and for all integers $n, m \geq 1$, $a^{mn} = (a^m)^n$. You may assume for any real numbers α and β:

a. $\alpha^1 = \alpha$,
b. $\alpha^i \alpha^j = \alpha^{i+j}$, for all integers $i, j \geq 1$,
c. $\alpha^i \beta^i = (\alpha\beta)^i$, for all integers $i \geq 1$.

(Hint: Fix $n \geq 1$.)

Fix $n \geq 1$. For $m \geq 1$, let $P(m) =$ "$a^{mn} = (a^m)^n$".
Basis step: $P(1)$ is true since $a^{1 \cdot n} = a^n = (a^1)^n$.
Inductive step on m: For $m \geq 1$, $P(m) \Rightarrow P(m+1)$, since if $a^{mn} = (a^m)^n$ then

$$\begin{aligned} a^{(m+1)n} &= a^{mn+n} = a^{mn}a^n \\ &= (a^m)^n a^n \\ &= (a^m a)^n \\ &= (a^{m+1})^n. \end{aligned}$$

8. [10] Prove for any sets A, B, C, and D that $(A \cap B) \sim (C \cap D) \subseteq (A \sim C) \cup (B \sim D)$.

We have

$$\begin{aligned} & x \in (A \cap B) \sim (C \cap D) \\ & \Rightarrow (x \in A \cap B) \wedge \sim (x \in C \cap D) \\ & \Rightarrow (x \in A \wedge x \in B) \wedge \sim (x \in C \wedge x \in D) \\ & \Rightarrow (x \in A \wedge x \in B \wedge x \notin C) \vee (x \in A \wedge x \in B \wedge x \notin D) \\ & \Rightarrow (x \in A \wedge x \notin C) \vee (x \in B \wedge x \notin D) \\ & \Rightarrow x \in (A \sim C) \cup (B \sim D). \end{aligned}$$

9. [10]. Given sets A, B, and C. Prove that $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

We have

$$\begin{aligned} & (x, y) \in (A \times B) \cup (C \times D) \\ & \Rightarrow (x \in A \wedge y \in B) \vee (x \in C \wedge y \in D) \\ & \Rightarrow ((x \in A \wedge y \in B) \vee x \in C) \wedge ((x \in A \wedge y \in B) \vee y \in D) \\ & \Rightarrow ((x \in A \vee x \in C) \wedge (y \in B \vee x \in C)) \wedge ((x \in A \vee y \in D) \wedge (y \in B \vee y \in D)) \\ & \Rightarrow (x \in A \vee x \in C) \wedge (y \in B \vee y \in D)) \\ & \Rightarrow (x, y) \in (A \cup C) \times (B \cup D). \end{aligned}$$

10. Let R be a relation on a non-empty set A. Define $R^1 = R$ and $R^{n+1} = R \circ R^n$ for $n \geq 1$.

a. [5]. Prove (from the definition) that if R is transitive $R^2 \subseteq R$.

Take $(x, y) \in R^2$. By definition of R^2, there exists a $z \in A$ so that $(x, z) \in R$ and $(z, y) \in R$. But by transitivity, $(x, y) \in R$ so $R^2 \subseteq R$.

b. [10]. Prove that if R is transitive $R^n \subseteq R$ for all $n \geq 1$. (You may assume that if $S_1 \subseteq S_2$ and $T_1 \subseteq T_2$ then $S_1 \circ T_1 \subseteq S_2 \circ T_2$.)

We proceed by induction. For $n = 1$, $R^1 = R \subseteq R$. Now assume for some $n \geq 1$, that if R is transitive then $R^n \subseteq R$. But then $R^{n+1} = R \circ R^n \subseteq R \circ R = R^2 \subseteq R$.

11. a.[10] Given a function $f : A \to B$, let E be a relation on A defined by $(x, y) \in E$ if and only if $f(x) = f(y)$. Prove that E is an equivalence relation.

We must prove that E is reflexive, symmetric, and transitive. Since for all $x \in A$, $f(x) = f(x)$, we have $(x, x) \in E$ so E is reflexive. Next, since if $(x, y) \in E$ then $f(x) = f(y)$ and $f(y) = f(x)$ so $(y, x) \in E$ and E is symmetric. Finally, if $(x, y) \in E$ and $(y, z) \in E$ then $f(x) = f(y)$ and $f(y) = f(z)$ so $f(x) = f(z)$ and $(x, z) \in E$, so E is transitive. Since E is reflexive, symmetric, and transitive it is an equivalence relation.

b. (5) Let $A = \{-10, \ldots -1, 0, 1, 2, \ldots, 10\}$, $B = \mathbb{N}$ and $f(x) = x^2$. Specify the elements of the partition that E imposes on A. (Hint: Recall elements of the partition are sets.)

The partition is:
$\{\{-10,10\},\{-9,9\},\{-8,8\},\{-7,7\},\{-6,6\},(-5,5\},\{-4,4\},\{-3,3\},\{-2,2\},\{-1,1\},\{0\}\}$

12. Given a non-empty set A and function $f : A \to A$,

a. [10] Prove that if $f \circ f$ is one-to-one then f is one-to-one.

Given $x, y \in A$, suppose $f(x) = f(y)$. Then $f \circ f(x) = f(f(x)) = f(f(y)) = f \circ f(y)$. But since $f \circ f$ is one-to-one then $x = y$. We conclude f is one-to-one.

b. [10] Prove that if $f \circ f$ is onto then f is onto.

Since $f \circ f$ is onto, for any $z \in A$ there exists $x \in A$ so that $f \circ f(x) = z$. but since $f \circ f(x) = f(f(x))$, letting $y = f(x)$, we have an element $y \in A$ so that $f(y) = z$. We conclude f is onto.

13. [15] For any $n \geq 1$, consider the set $B = \{1, 2, ..., 2n\}$. Prove that if $A \subseteq B$ and $|A| \geq n+1$ then there exist $a, b \in A$ so that $a + b = 2n+1$. (Hint consider a function $f(x) = \begin{cases} x & x \leq n \\ 2n+1-x & x \geq n+1 \end{cases}$.)

The function f maps A into $\{1, ..., n\}$. By the pigeonhole principle there exist distinct $a, b \in A$ so that $f(a) = f(b)$. Since a and b are distinct we cannot have both $a, b \leq n$ since then $a = f(a) = f(b) = b$. Similarly we cannot have both $a, b \geq n+1$ since then $2n+1-a = f(a) = f(b) = 2n+1-b$ and again $a = b$. Thus, without loss of generality, assume $a \leq n$ and $b \geq n+1$. Then since $a = f(a) = f(b) = 2n+1-b$, we have $a + b = 2n+1$.