Final Examination Solutions

CS 313H

1. [10] Use a truth table to determine for which truth values of $p, q,$ and r $(p \vee r) \Rightarrow (q \wedge r)$ is true.

p	q	r	$p \vee r$	$q \wedge r$	$(p \vee r) \Rightarrow (q \wedge r)$
F	F	F	F	F	T
F	F	T	T	F	F
F	T	F	F	F	T
F	T	T	T	T	T
T	F	F	T	F	F
T	F	T	T	F	F
T	T	F	T	F	F
T	T	T	T	T	T

The expression $(p \vee r) \Rightarrow (q \wedge r)$ is true when q is false (except when p is also true) and when all of $p, q,$ and r are true.

2. [20] Using sentential calculus (with a four column format), prove that the conclusion $s \wedge \sim s$ follows from premises: $p \Rightarrow q, \sim (q \vee r),$ and $r \Leftrightarrow \sim p$.

$\{Pr_1\}$	(1.) $p \Rightarrow q$	P
$\{Pr_2\}$	(2.) $\sim (q \vee r)$	P
$\{Pr_3\}$	(3.) $r \Leftrightarrow \sim p$	P
$\{Pr_2\}$	(4.) $\sim q \wedge \sim r$	DeM (2)
$\{Pr_2\}$	(5.) $\sim q$	Simp (5)
$\{Pr_2\}$	(6.) $\sim r$	Simp (5)
$\{Pr_3\}$	(7.) $(r \Rightarrow \sim p) \wedge (\sim r \Rightarrow \sim\sim p)$	Taut (3)
$\{Pr_3\}$	(8.) $\sim r \Rightarrow \sim\sim p$	Simp (7)
$\{Pr_2, Pr_3\}$	(9.) $\sim\sim p$	MP (8)
$\{Pr_2, Pr_3\}$	(10.) p	DN (9)
$\{Pr_1, Pr_2, Pr_3\}$	(11.) q	MP (1), (10)
$\{Pr_1, Pr_2, Pr_3\}$	(12.) $s \wedge \sim s$	Clav (6), (13)

3. [20] Prove that the conclusion $q \wedge r$ follows from the premises $p \Rightarrow q, q \vee r, p \Rightarrow r, q \Rightarrow p,$ and $r \Rightarrow p$. First convert the premises and the negation of the conclusion into Conjunctive Normal Form, and then employ a resolution proof to get a contradiction.

$p \Rightarrow q$
$\sim p \vee q$

$q \vee r$

$p \Rightarrow r$
$\sim p \vee r$

$q \Rightarrow p$
$\sim q \vee p$

$r \Rightarrow p$
$\sim r \vee p$

$\sim (q \wedge r)$
$\sim q \vee \sim r$

1. $\sim p \vee q$	P
2. $q \vee r$	P
3. $\sim p \vee r$	P
4. $\sim q \vee p$	P
5. $\sim r \vee p$	P
6. $\sim q \vee \sim r$	P
7. $r \vee p$	Res (2), (4)
8. p	Res (5), (7)
9. q	Res (1), (8)
10. $\sim r$	Res (6), (9)
11. $\sim p$	Res (3), (10)
9. *false*	Conj. (8), (11)

4. [10] Using the predicates defined on the set $\mathbb{Z}$ of integers :

Px	x is prime,
Ox	x is odd,
Exy	x equal to y
Lxy	x is less than y.

Express in the syntax of Predicate Calculus (you may use any integers as constants):

a. *All prime numbers are positive and the only even prime number is 2.*

$$(\forall x \in \mathbb{Z})(Px \Rightarrow L0x) \wedge (\forall x \in \mathbb{Z})((Px \wedge \sim Ox) \Leftrightarrow Ex2)$$

b. *There is no prime number strictly between 317 and 337.*

$$(\forall x \in \mathbb{Z})((L317x \wedge Lx337) \Rightarrow \sim Px)$$

5. [25] Prove that $(\forall x)Px \wedge (\forall x)Qx$ follows from $(\forall x)(Px \wedge Qx)$ (Rather than using the TC rule be specific about the sentential calculus rule.)

$\{P_1\}$	(1). $(\forall x)(Px \wedge Qx)$	P
$\{P_1\}$	(2). $Pa \wedge Qa$	UI (1)
$\{P_1\}$	(3). Pa	Simp (2)
$\{P_1\}$	(4). $(\forall x)Px$	EG (3)
$\{P_1\}$	(5). Qa	Simp (2)
$\{P_1\}$	(6). $(\forall x)Qx$	EG (3)
$\{P_1\}$	(7). $(\forall x)Px \wedge (\forall x)Qx$	Conj (6)

6. [10] Using induction, prove that for $r \neq 0,1$ and $n \geq 0$, $\sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1}$.

For $n \geq 0$, let $P(n)=$ "$\sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1}$".

Basis step: $P(0)$ is true since $\sum_{k=0}^{0} r^k = 1 = \frac{r^{0+1}-1}{r-1}$..

Inductive step: For $n \geq 1$, $P(n) \Rightarrow P(n+1)$, since if $\sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1}$, then

$$\begin{aligned}\sum_{k=0}^{n+1} r^k &= \sum_{i=0}^{n} r^k + r^{n+1} \\ &= \frac{r^{n+1}-1}{r-1} + \frac{r^{n+2}-r^{n+1}}{r-1} \\ &= \frac{r^{(n+1)+1}-1}{r-1}.\end{aligned}$$

7. [10] Consider the Fibonacci sequence: $f_0 = 1, f_1 = 1, f_k = f_{k-1} + f_{k-2}$, for $k \geq 2$. Using induction, prove that for $n \geq 0$, $\sum_{k=0}^{n} f_k^2 = f_n f_{n+1}$.

For $n \geq 0$, let $P(n)$ = "$\sum_{k=0}^{n} f_k^2 = f_n f_{n+1}$".

Basis step: $P(0)$ is true since $\sum_{k=0}^{n} f_k^2 = f_0^2 = 1 = 1 \cdot 1 = f_0 f_1$..

Inductive step: For $n \geq 0$, $P(n) \Rightarrow P(n+1)$, since if $\sum_{k=0}^{n} f_k^2 = f_n f_{n+1}$, then

$$\begin{aligned}\sum_{k=0}^{n+1} f_k^2 &= \sum_{k=0}^{n} f_k^2 + f_{n+1}^2 \\ &= f_n f_{n+1} + f_{n+1}^2 \\ &= f_{n+1}(f_n + f_{n+1}) \\ &= f_{n+1} f_{(n+1)+1}.\end{aligned}$$

8. [10] Prove for any sets $A, B, C,$ and D that $(A \cup B) \sim (C \cup D) \subseteq (A \sim C) \cup (B \sim D)$.

We have

$$\begin{aligned}&x \in (A \cup B) \sim (C \cup D) \\ &\Rightarrow (x \in A \cup B) \wedge \sim (x \in C \cup D) \\ &\Rightarrow (x \in A \vee x \in B) \wedge \sim (x \in C \vee x \in D) \\ &\Rightarrow (x \in A \vee x \in B) \wedge (x \notin C \wedge x \notin D) \\ &\Rightarrow (x \in A \wedge x \notin C \wedge x \notin D) \vee (x \in B \wedge x \notin C \wedge x \notin D) \\ &\Rightarrow (x \in A \wedge x \notin C) \vee (x \in B \wedge x \notin D) \\ &\Rightarrow (x \in A \sim C) \vee (x \in B \sim D) \\ &\Rightarrow x \in (A \sim C) \cup (B \sim D).\end{aligned}$$

9. [10]. Given sets $A, B,$ and C be sets. Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

$$(x, y) \in A \times (B \cup C)$$
$$\Leftrightarrow x \in A \wedge y \in B \cup C$$
$$\Leftrightarrow x \in A \wedge (y \in B \vee y \in C)$$
$$\Leftrightarrow (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C)$$
$$\Leftrightarrow (x, y) \in (A \times B) \vee (x, y) \in (A \times C)$$
$$\Leftrightarrow (x, y) \in (A \times B) \cup (A \times C).$$

10. **[15]**. Let R be defined $R = \{(x, y) : y \text{ is an integer multiple of } x\}$. Prove that R is a partial order on $\mathbb{Z}^+$.

We need to show that R is reflexive, antisymmetric, and transitive. For any $x \in \mathbb{Z}^+$ x is an integer multiple of x so $(x, x) \in R$ and R is reflexive. Next suppose $(x, y), (y, x) \in R$ so y is an integer multiple of x and x is an integer multiple of y . We have then for some integers k and j, $y = kx$ and $x = jy$. But then $y = kjy$ so $kj = 1$ and since k and j are integers, both are one. We conclude that $x = y$ and R is antisymmetric. Lastly, if $(x, y) \in R$ and $(y, z) \in R$ then for some integers k and j, $x = ky$ and $y = jz$ so $x = kjz$, $(x, z) \in R$, and R is transitive. We conclude R is a partial order on $\mathbb{Z}^+$.

11. [10] Let A be any set and R be a relation on A. Prove that if R is both symmetric and antisymmetric then must also be transitive.

We must prove that $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$. To that end suppose $(x, y) \in R$ and $(y, z) \in R$. By symmetry both $(y, x) \in R$ and $(z, y) \in R$. By antisymmetry $x = y$ and $y = z$, thus $x = y = z$. So, $(x, y) \in R$ means $(x, x) \in R$ and this is the same as $(x, z) \in R$.

12. [10] Given $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ and $g : \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by $f(n) = n!$ and $g(n) = n - 1$, respectively, what are

a. $f \circ f$

$f \circ f(n) = f(f(n)) = (n!)!.$

b. $f \circ g$

$f \circ g(n) = f(g(n)) = (n-1)!.$

c. $g \circ f$

$g \circ f(n) = g(f(n)) = n! - 1.$

d. $g \circ g$

$g \circ g(n) = g(g(n)) = (n-1) - 1 = n - 2.$

13. Given a sets $A, B,$ and C and functions $f : A \to B$ and $g : B \to C$,

a. [10] Prove that if $g \circ f$ is one-to-one then f is one-to-one.

Given $x, y \in A$, suppose $f(x) = f(y)$. Then $g \circ f(x) = g(f(x)) = g(f(y)) = g \circ f(y)$. But since $g \circ f$ is one-to-one then $x = y$. We conclude f is one-to-one.

b. [10] Prove that if $g \circ f$ is onto then g is onto.

Since $g \circ f$ is onto, for any $z \in C$ there exists $x \in A$ so that $g \circ f(x) = z$. but since $g \circ f(x) = g(f(x))$, letting $y = f(x)$, we have an element $y \in B$ so that $g(y) = z$. We conclude g is onto.