Practice Final Examination Solutions

CS 313H

1. [10] Use a truth table to determine for which truth values of $p, q,$ and r $\sim(p \wedge r) \vee (\sim q \wedge r)$ is true.

p	q	r	$p \wedge r$	$\sim(p \wedge r)$	$\sim q$	$\sim q \wedge r$	$\sim(p \wedge r) \vee (\sim q \wedge r)$
F	F	F	F	T	T	F	T
F	F	T	F	T	T	T	T
F	T	F	F	T	F	F	T
F	T	T	F	T	F	F	T
T	F	F	F	T	T	F	T
T	F	T	T	F	T	T	T
T	T	F	F	T	F	F	T
T	T	T	T	F	F	F	F

The expression $\sim(p \wedge r) \vee (\sim q \wedge r)$ is true for all truth values of $p, q,$ and r except all of $p, q,$ and r being true.

2. [20] Using sentential calculus (with a four column format), prove that the conclusion p follows from premises: $p \vee q, q \Rightarrow t, \sim r \vee \sim s, (s \wedge t) \Rightarrow r,$ and $q \Rightarrow s$.

$\{Pr_1\}$	(1.) $p \vee q$	P
$\{Pr_2\}$	(2.) $q \Rightarrow t$	P
$\{Pr_3\}$	(3.) $\sim r \vee \sim s$	P
$\{Pr_4\}$	(4.) $(s \wedge t) \Rightarrow r$	P
$\{Pr_5\}$	(5.) $q \Rightarrow s$	P
$\{Pr_6\}$	(6.) $\sim p$	P (for CP)
$\{Pr_1, Pr_6\}$	(7.) q	DS, (1), (6)
$\{Pr_1, Pr_2, Pr_6\}$	(8.) t	MP (2), (7)
$\{Pr_1, Pr_5, Pr_6\}$	(9.) s	MP (5), (7)
$\{Pr_1, Pr_2, Pr_5, Pr_6\}$	(10.) $s \wedge t$	Conj. (8), (9)
$\{Pr_1, Pr_2, Pr_4, Pr_5, Pr_6\}$	(11.) r	MP (4), (10)
$\{Pr_1, Pr_2, Pr_3, Pr_4, Pr_5, Pr_6\}$	(12.) $\sim s$	DS, (3), (11)
$\{Pr_1, Pr_2, Pr_3, Pr_4, Pr_5, Pr_6\}$	(13.) p	ContraPrm. (9), (12)
$\{Pr_1, Pr_2, Pr_3, Pr_4, Pr_5\}$	(14.) $\sim p \Rightarrow p$	C (6), (13)
$\{Pr_1, Pr_2, Pr_3, Pr_4, Pr_5\}$	(15.) p	Clav (6), (13)

3. [20] Prove that the conclusion p follows from the premises $((p \Rightarrow q) \wedge (p \wedge \sim q)) \vee r$ and $r \Rightarrow p$. First convert the premises and the negation of the conclusion into Conjunctive Normal Form, and then employ a resolution proof to get a contradiction.

$$((p \Rightarrow q) \wedge (p \wedge \sim q)) \vee r$$
$$((\sim p \vee q) \wedge (p \wedge \sim q)) \vee r$$
$$((\sim p \vee q) \vee r) \wedge ((p \wedge \sim q) \vee r)$$
$$(\sim p \vee q \vee r) \wedge (p \vee r) \wedge (\sim q \vee r)$$

$$r \Rightarrow p$$
$$\sim r \vee p$$

$$\sim p$$

1.	$\sim p \vee q \vee r$	P
2.	$p \vee r$	P
3.	$\sim q \vee r$	P
4.	$\sim r \vee p$	P
5.	$\sim p$	P
6.	$\sim r$	Res (4), (5)
7.	p	Res (2), (6)
8.	*false*	Conj. (5), (7)

4. [10] Using the predicates defined on the set L of upper case Latin characters, the set $\square^+$ of positive integers, and the set S of strings of upper case Latin characters :

Vx	x is a vowel, for $x \in L$,
Sxn	x can be written in n strokes, for $x \in L$ and $n \in \square^+$,
Wxs	x occurs in the string s, for $x \in L$ and $s \in S$,
Bxy	x occurs before y in the English alphabet, for $x, y \in L$.
Exy	x equals y, for $x, y \in L$.

Express in the syntax of Predicate Calculus (you may use upper case Latin characters, positive integers, and strings of upper case Latin characters as constants):

a. *'A' is the only upper case Latin character that is a vowel and can be written in three strokes but does not occur in the string 'STUPID'.*

$$(\forall x \in L)((Vx \wedge Sx3 \wedge \sim Wx'STUPID') \Leftrightarrow (ExA))$$

b. *There is an upper case Latin character strictly between 'K' and 'R' that can be written in one stroke.*

$$(\exists x \in L)(BKx \wedge BxR \wedge Sx1)$$

5. [25] Prove that $(\forall u)(\exists v)Rvu$ follows from $(\exists x)(\forall y)Rxy$ (Rather than using the TC rule be specific about the sentential calculus rule.)

$\{P_1\}$	(1). $(\exists x)(\forall y)Rxy$	P
$\{P_1\}$	(2). $(\forall y)Ray$	EI (1)
$\{P_1\}$	(3). Rab	UI (2)
$\{P_1\}$	(4). $(\exists v)Rvb$	EG (3)
$\{P_1\}$	(5). $(\forall u)(\exists v)Rvu$	UG (4)

6. [10] Using induction, prove that for $n \geq 1$, $\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$.

For $n \geq 1$, let $P(n)$ = "$\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1.$".

Basis step: $P(1)$ is true since $\sum_{k=1}^{1} k \cdot k! = 1 \cdot 1! = 1 = 2 - 1 = (1+1)! - 1.$.

Inductive step: For $n \geq 1$, $P(n) \Rightarrow P(n+1)$, since if $\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1.$, then

$$\begin{aligned}\sum_{k=1}^{n+1} k \cdot k! &= \sum_{i=0}^{n} k \cdot k! + (n+1)(n+1)! \\ &= (n+1)! - 1 + (n+1)(n+1)! \\ &= (n+1+1)(n+1)! - 1.\end{aligned}$$

7. [10] a. Given a sequence of integers $a_1, a_2, \ldots$ such that $a_k > a_{k-1}$ for $k \geq 2$, using induction to prove that for $k \geq 1$, $a_k \geq a_1 + k - 1$. (Notice $a_k > a_{k-1}$ is equivalent to $a_k \geq a_{k-1} + 1$.)

For $k \geq 1$, let $P(k)$ = "$a_k \geq a_1 + k - 1$".

Basis step: $P(1)$ is true since $a_1 \geq a_1 + (1-1)$.

Inductive step: For $k \geq 1$, $P(k) \Rightarrow P(k+1)$, since if $a_k \geq a_1 + k - 1$, then

$$\begin{aligned}a_{k+1} &= (a_{k+1} - a_k) + a_k \\ &\geq a_1 + k - 1 + (a_{k+1} - a_k) \\ &\geq a_1 + k - 1 + 1 \\ &\geq a_1 + (k+1) - 1.\end{aligned}$$

b. [5] Using this, prove that for any integer m, display a k so that $a_k \geq m$.

Given any integer m, let $k = \max\{1, m+1-a_1\}$ then $k \geq 1$ and $k \geq m+1-a_1$ so $a_k \geq a_1 + k - 1 \geq a_1 + m + 1 - a_1 - 1 = m$.

8. [10] Prove for any sets $A, B,$ and C that $A \sim (B \sim C) = (A \sim B) \cup (A \cap C)$.

We have

$$
\begin{aligned}
& x \in A \sim (B \sim C) \\
& \Leftrightarrow x \in A \wedge \sim (x \in B \sim C) \\
& \Leftrightarrow x \in A \wedge \sim (x \in B \wedge x \notin C) \\
& \Leftrightarrow x \in A \wedge (x \notin B \vee x \in C) \\
& \Leftrightarrow (x \in A \wedge x \notin B) \vee (x \in A \wedge x \in C) \\
& \Leftrightarrow (x \in A \sim B) \vee (x \in A \cap C) \\
& \Leftrightarrow x \in (A \sim B) \cup (A \cap C)
\end{aligned}
$$

9. [10]. Given sets $A, B, C,$ and D be sets. Prove that $A \times B \subseteq C \times D$ if and only if $A \subseteq C \wedge B \subseteq D.$

Suppose $A \subseteq C \wedge B \subseteq D,$ then

$$
\begin{aligned}
& (x, y) \in A \times B \\
& \Rightarrow x \in A \wedge y \in B \\
& \Rightarrow x \in C \wedge y \in D \\
& \Rightarrow (x, y) \in C \times D.
\end{aligned}
$$

Suppose $A \times B \subseteq C \times D,$ then

$$
\begin{aligned}
& x \in A \wedge y \in B \\
& \Rightarrow (x, y) \in A \times B \\
& \Rightarrow (x, y) \in C \times D \\
& \Rightarrow x \in C \wedge y \in D,
\end{aligned}
$$

Thus $A \subseteq C \wedge B \subseteq D,$

10. **[15]**. Let R be defined $R = \{((x, y), (u, v)) : x^2 + y^2 = u^2 + v^2\}$. Prove that R is an equivalence relation on $\mathbb{R}^2$.

We need to show that R is reflexive, symmetric, and transitive. For any $(x, y) \in \mathbb{R}^2$ $x^2 + y^2 = x^2 + y^2$ so $((x, y), (x, y)) \in R$ and R is reflexive. Next since for $(x, y), (u, v) \in \mathbb{R}^2$ if $x^2 + y^2 = u^2 + v^2$ then $u^2 + v^2 = x^2 + y^2$ so $((x, y), (u, v)) \in R$ implies $((u, v), (x, y)) \in R$ is symmetric. Lastly, if $((x, y), (u, v)) \in R$ and $((u, v), (w, z)) \in R$ then $x^2 + y^2 = u^2 + v^2$ and $u^2 + v^2 = w^2 + z^2$ so $x^2 + y^2 = w^2 + z^2$, $((x, y), (w, z)) \in R$, and R is transitive. We conclude R is an equivalence relation on $\mathbb{R}^2$.

11.. Consider a relation R on a set A . Prove or disprove with a simple counter example each of the following:

a. [10] If R is reflexive, then R^2 is reflexive.

For all $x \in A$, since R is reflexive then $(x,x) \in R$. But then $(x,x) \in R$ and $(x,x) \in R$ imply $(x,x) \in R^2$ so R^2 is reflexive.

b. [10] If R is symmetric, then R^2 is symmetric.

For all $x, y \in A$, if $(x, y) \in R^2$ then for some $z \in A, (x,z) \in R$ and $(z, y) \in R$. But then since R is symmetric for the same $z, (z,x) \in R$ and $(y,z) \in R$ so then $(y,x) \in R^2$ so R^2 is symmetric.

c. [10] If R is antisymmetric, then R^2 is antisymmetric.

Let $A = \{1,2,3\}$ and $R = \{(1,2),(2,3),(3,1),(3,3)\}$ so $R^2 = \{(1,3),(2,3),(2,1),(3,2),(3,1)\}$. R is antisymmetric, but R^2 is not antisymmetric since both $(2,3),(3,2) \in R^2$.

d. [10] If R is transitive, then R^2 is transitive.

For all $x, y \in A$, if $(x, y),(y,z) \in R^2$ then for some $z \in A, (x,z) \in R$ and $(z, y) \in R$ and for some $w \in A, (y, w) \in R$ and $(w,z) \in R$. Since R is transitive $(x, y) \in R$ and $(y,z) \in R$ so then $(x, y) \in R^2$ so R^2 is transitive.

12. [10] Given $f : \square^+ \to \square^+$ and $g : \square^+ \to \square^+$ defined by $f(n) = n^2$ and $g(n) = 2^n$, respectively, what are

a. $f \circ f$

$f \circ f(n) = f(f(n)) = (n^2)^2 = n^4$.

b. $f \circ g$

$f \circ g(n) = f(g(n)) = (2^n)^2 = 2^{2n}$.

c. $g \circ f$

$g \circ f(n) = g(f(n)) = 2^{(n^2)}$.

d. $g \circ g$

$g \circ g(n) = g(g(n)) = 2^{(2^n)}$.

13. [20] Given a sets A and B and function $f : A \to B$, Prove that f is one-to-one if and only if $f(X \cap Y) = f(X) \cap f(Y)$ for all $X, Y \subseteq A$. (Hint: Recall $f(\varnothing) = \varnothing$ and $f(\{x\}) = \{f(x)\}$ for any $x \in A$.)

We will first show that if $f(X \cap Y) = f(X) \cap f(Y)$ then f is one-to-one. To that end, suppose $f(X \cap Y) = f(X) \cap f(Y)$ and $f(x) = f(y)$. Then $f(\{x\}) = f(\{y\})$ and $\varnothing \neq f(\{x\}) = f(\{x\}) \cap f(\{y\}) = f(\{x\} \cap \{y\})$ so $\{x\} \cap \{y\}$ is nonempty, but this means $x = y$ so f is one-to-one.

Next we will show that if f is one-to-one then $f(X) \cap f(Y) \subseteq f(X \cap Y)$. Take any $z \in f(X) \cap f(Y)$ thus $z \in f(X)$ and $z \in f(Y)$, so for some $x \in X, f(x) = z$ and for some $y \in Y, f(y) = z$. But f is one-to-one, so $x = y$. Since $x = y$ and $y \in Y$ we have $x \in Y$ and $x \in X \cap Y$. Since $f(x) = z, z \in X \cap Y$ and we have proved that $f(X) \cap f(Y) \subseteq f(X \cap Y)$.

Lastly, we show that $f(X \cap Y) \subseteq f(X) \cap f(Y)$. (This is actually true whether f is one-to-one or not.) . Take any $z \in f(X \cap Y)$ thus for some $x \in X \cap Y, f(x) = z$ and for some $y \in Y, f(y) = z$. Since $x \in X \cap Y, z \in f(X)$ and $z \in f(Y)$, so $z \in f(X) \cap f(Y)$ and $f(X \cap Y) \subseteq f(X) \cap f(Y)$.