Final Examination Solutions

CS 313H

1a. [5] Use a truth table to determine for which truth values of $p, q,$ and r $(p \vee \sim r) \Rightarrow (\sim q \wedge p)$ is true.

p	q	r	$\sim r$	$\sim q$	$p \vee \sim r$	$\sim q \wedge p$	$(p \vee \sim r) \Rightarrow (\sim q \wedge p)$
F	F	F	T	T	T	F	F
F	F	T	F	T	F	F	T
F	T	F	T	F	T	F	F
F	T	T	F	F	F	F	T
T	F	F	T	T	T	T	T
T	F	T	F	T	T	T	T
T	T	F	T	F	T	F	F
T	T	T	F	F	T	F	F

1a. [5] Use the information to construct a logical expression $exp(p,q,r)$ so that you have

$$exp(p,q,r) = (\sim p \wedge r) \vee (p \wedge \sim q)$$

2. [20] Using sentential calculus (with a four column format), prove that the conclusion $\sim p \wedge \sim q$ follows from premises: $(p \Rightarrow r) \wedge (q \Rightarrow s), (s \wedge r) \Rightarrow t, \sim t$ and $\sim (\sim p \vee \sim q)$.

$\{Pr_1\}$	(1.) $(p \Rightarrow r) \wedge (q \Rightarrow s)$	P
$\{Pr_2\}$	(2.) $(s \wedge r) \Rightarrow t$	P
$\{Pr_3\}$	(3.) $\sim t$	P
$\{Pr_4\}$	(4.) $\sim (\sim p \vee \sim q)$	P
$\{Pr_4\}$	(5.) $\sim\sim p \wedge \sim\sim q$	DeM (4)
$\{Pr_4\}$	(6.) $p \wedge q$	DN (5)
$\{Pr_4\}$	(7.) p	Simp (6)
$\{Pr_1\}$	(8.) $p \Rightarrow r$	Simp (1)
$\{Pr_1, Pr_4\}$	(9.) r	MP (7), (8)
$\{Pr_4\}$	(10.) q	Simp (6)
$\{Pr_1\}$	(11.) $q \Rightarrow s$	Simp (1)
$\{Pr_1, Pr_4\}$	(12.) s	MP (10), (11)
$\{Pr_1, Pr_4\}$	(13.) $s \wedge r$	Add (9), (12)
$\{Pr_1, Pr_2, Pr_4\}$	(14.) t	MP (13), (2)
$\{Pr_1, Pr_2, Pr_3, Pr_4\}$	(15.) $t \wedge \sim t$	Add (14), (3)
$\{Pr_1, Pr_2, Pr_3, Pr_4\}$	(16.) $\sim p \wedge \sim q$	Clav (15)

3. [15] Prove that the conclusion $\sim q \wedge p$ follows from the premises $(p \Rightarrow q) \Rightarrow r$ and $\sim r$. First convert the premise and the negation of the conclusion into Conjunctive Normal Form, and then employ a resolution proof to get a contradiction.

$(p \Rightarrow q) \Rightarrow r$
$\sim (p \Rightarrow q) \vee r$
$\sim (\sim p \vee q) \vee r$
$(\sim\sim p \wedge \sim q) \vee r$
$(p \wedge \sim q) \vee r$
$(p \vee r) \wedge (\sim q \vee r)$

$\sim (\sim q \wedge p)$
$\sim\sim q \vee \sim p$
$q \vee \sim p$

1.	$p \vee r$	P
2.	$\sim q \vee r$	P
3.	$q \vee \sim p$	P
4.	$\sim r$	P
5.	$q \vee r$	Res (1), (3)
6.	r	Res (2), (5)
7.	*false*	Conj. (4), (6)

4. [10] Using the predicates defined on the set P of persons:

Mx	x is a male,
Fx	x is a female,
Exy	x equal to y
Txy	x is strictly taller than y.

Express in the syntax of Predicate Calculus:

a. *The tallest person is a male.*

$$(\forall x)(Mx \wedge (\forall y)(\sim Exy \Rightarrow Txy)$$

b. *Some male persons is strictly taller than all female persons but some female person is taller than some male person.*

$$(\exists x)(Mx \wedge (\forall y)(Fy \Rightarrow Txy)) \wedge (\exists y)(Fy \wedge (\exists x)(Mx \wedge Tyx))$$

5. [25] Prove that $(\forall x)Px \wedge (\forall x)Qx$ follows from $(\forall x)(Px \wedge Qx)$ (Rather than using the TC rule be specific about the sentential calculus rule.)

$\{P_1\}$	(1).	$(\forall x)(Px \wedge Qx)$	P
$\{P_1\}$	(2).	$Pa \wedge Qa$	UI (1)
$\{P_1\}$	(3).	Pa	Simp (2)
$\{P_1\}$	(4).	$(\forall x)Px$	EG (3)
$\{P_1\}$	(5).	Qa	Simp (2)
$\{P_1\}$	(6).	$(\forall x)Qx$	EG (3)
$\{P_1\}$	(7).	$(\forall x)Px \wedge (\forall x)Qx$	Conj (6)

6a. [5] Using induction, prove that for $n \geq 0, 1 \leq 3^n$.

For $n \geq 0$, let $P(n) =$ "$1 \leq 3^n$".

Basis step: $P(0)$ is true since $1 \leq 1 = 3^0$..

Inductive step: For $n \geq 0$, $P(n) \Rightarrow P(n+1)$, since if $1 \leq 3^n$, then

$$
\begin{aligned}
1 &\leq 3 \\
&\leq 3 \cdot 3^n \\
&\leq 3^{n+1}.
\end{aligned}
$$

b. [5] Using induction and part **a**, prove that for $n \geq 2$, $1 + 2n < 3^n$.

For $n \geq 2$, let $P(n)$ = "$1 + 2n < 3^n$".

Basis step: $P(2)$ is true since $1 + 2 \cdot 2 = 5 < 9 = 3^2$..

Inductive step: For $n \geq 0$, $P(n) \Rightarrow P(n+1)$, since if $1+2n < 3^n$, then

$$\begin{aligned} 1+2(n+1) &= 1+2n+2 \\ &< 3^n + 2 \\ &< 3^n + 2\cdot 1 \\ &< 3^n + 2\cdot 3^n \\ &< 3^{n+1}. \end{aligned}$$

7. [10] Consider the sequence: $a_0 = 2, a_1 = 1, a_n = a_{n-1} + 2a_{n-2}$, for $n \geq 2$. Using induction, prove that for $n \geq 0$, $a_n = 2^n + (-1)^n$.

For $n \geq 0$, let $P(n)$= "$a_n = 2^n + (-1)^n$".

Basis step: $P(0)$ and $P(1)$ are true since $a_0 = 2 = 1+1 = 2^0 + (-1)^0$ and $a_1 = 1 = 2-1 = 2^1 + (-1)^1$.

Inductive step: For $n \geq 2$, $P(n) \Rightarrow P(n+1)$, since if $a_{n-2} = 2^{n-2} + (-1)^{n-2}$ and $a_{n-1} = 2^{n-1} + (-1)^{n-1}$, then

$$\begin{aligned} a_n &= a_{n-1} + 2a_{n-2} \\ &= 2^{n-1} + (-1)^{n-1} + 2(2^{n-2} + (-1)^{n-2}) \\ &= 2^{n-1} - (-1)^n + 2^{n-1} + 2(-1)^n \\ &= 2\cdot 2^{n-1} + (-1)^n \\ &= 2^n + (-1)^n. \end{aligned}$$

8. [10] Prove for any sets A, B, C, and D that $(A \cap B) \sim (C \cup D) \subseteq (A \sim C) \cap (A \sim D)$.

We have

$$\begin{aligned} & x \in (A \cap B) \sim (C \cup D) \\ & \Rightarrow (x \in A \cap B) \wedge \sim (x \in C \cup D) \\ & \Rightarrow (x \in A \vee x \in B) \wedge \sim (x \in C \vee x \in D) \\ & \Rightarrow x \in A \wedge (x \notin C \wedge x \notin D) \\ & \Rightarrow (x \in A \wedge x \notin C) \wedge (x \in A \wedge x \notin D) \\ & \Rightarrow x \in (A \sim C) \cap (A \sim D). \end{aligned}$$

9. [10]. Given sets $A, B,$ and C be sets. Prove that $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

$$
\begin{aligned}
&(x, y) \in (A \times B) \cup (C \times D) \\
&\Rightarrow (x \in A \wedge y \in B) \vee (x \in C \wedge y \in D) \\
&\Rightarrow ((x \in A \wedge y \in B) \vee x \in C) \wedge ((x \in A \wedge y \in B) \vee y \in D) \\
&\Rightarrow ((x \in A \vee x \in C) \wedge (y \in B \vee x \in C)) \wedge ((x \in A \vee y \in D) \wedge (y \in B \vee y \in D)) \\
&\Rightarrow (x \in A \vee x \in C) \wedge (y \in B \vee y \in D)) \\
&\Rightarrow (x, y) \in (A \cup C) \times (B \cup D).
\end{aligned}
$$

10. Let R be a relation on a non-empty set A. Define $R^1 = R$ and $R^{n+1} = R \circ R^n$ for $n \geq 1$.

a. [5]. Prove (from the definition) that if R is transitive $R^2 \subseteq R$.

Take $(x, y) \in R^2$. By definition of R^2, there exists a $z \in A$ so that $(x, z) \in R$ and $(z, y) \in R$. But by transitivity, $(x, y) \in R$ so $R^2 \subseteq R$.

b. [10]. Prove that if R is transitive $R^n \subseteq R$ for all $n \geq 1$. (You may assume that if $S_1 \subseteq S_2$ and $T_1 \subseteq T_2$ then $S_1 \circ T_1 \subseteq S_2 \circ T_2$.)

We proceed by induction. For $n = 1$, $R^1 = R \subseteq R$. Now assume for some $n \geq 1$, that if R is transitive then $R^n \subseteq R$. But then $R^{n+1} = R \circ R^n \subseteq R \circ R = R^2 \subseteq R$.

11. [10] Let E be a relation on $\mathbb{R}^2$ (the Cartesian plane) defined by $((x, y), (u, v)) \in E$ if and only if $|x| + |y| = |u| + |v|$. Prove that E is an equivalence relation.

We must prove that E is reflexive, symmetric, and transitive. Since for all $(x, y) \in \mathbb{R}^2$, $|x| + |y| = |x| + |y|$, we have $((x, y), (x, y)) \in E$ so E is reflexive. Next, since if $((x, y), (u, v)) \in E$ then $|x| + |y| = |u| + |v|$ and $|u| + |v| = |x| + |y|$ so $((u, v), (x, y)) \in E$ and E is symmetric. Finally, if $((x, y), (u, v)) \in E$ and $((u, v), (w, z)) \in E$ then $|x| + |y| = |u| + |v|$ and $|u| + |v| = |w| + |z|$ so $|x| + |y| = |w| + |z|$ and $((x, y), (w, z)) \in E$, so E is transitive. Since E is reflexive, symmetric, and transitive it is an equivalence relation.

12. [10] Given $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$ and $g(x) = \sin(x)$, respectively, what are

a. $f \circ f$

$f \circ f(x) = f(f(x)) = e^{(e^x)}$.

b. $f \circ g$

$$f \circ g(x) = f(g(x)) = e^{\sin(x)}.$$

c. $g \circ f$

$$g \circ f(x) = g(f(x)) = \sin(e^x).$$

d. $g \circ g$

$$g \circ g(x) = g(g(x)) = (\sin(\sin(x)).$$

13. Given a non-empty set A and function $f: A \to A$, define $f^1 = f$ and $f^{n+1} = f \circ f^n$ for $n \geq 1$.

a. **[10]** Prove that if f is one-to-one then f^n is one-to-one for $n \geq 1$.

We prove this by induction. For $n = 1$, we have $f^1 = f$, so if f is one-to-one then f^1 is one-to-one. Now suppose if f is one-to-one then f^n is one-to-one and consider $f^{n+1} = f \circ f^n$. Given $x, y \in A$, suppose $f^{n+1}(x) = f^{n+1}(y)$. Then $f \circ f^n(x) = f(f^n(x)) = f(f^n(y)) = f \circ f^n(y)$. But since f is one-to-one, we have $f^n(x) = f^n(y)$ and then since f^n is one-to-one, we have $x = y$. We conclude f^{n+1} is one-to-one and so, by induction, if f is one-to-one then f^n is one-to-one for $n \geq 1$.

b. **[10]** Prove that if f is onto then f^n is onto for $n \geq 1$.

We prove this by induction. For $n = 1$, we have $f^1 = f$, so if f is onto then f^1 is onto. Now suppose if f is onto then f^n is onto and consider $f^{n+1} = f \circ f^n$. Since f is onto, given $y \in A$ there exists $x \in A$ such that $f(x) = y$ and since f^n is onto there exists a $z \in A$ so that $f^n(z) = x$. But this means there exists a $z \in A$ such that $f^{n+1}(z) = f(f^n(z)) = f(x) = y$. We conclude f^{n+1} is ono and so, by induction, if f is onto then f^n is onto for $n \geq 1$.