Final Examination

CS 313H

1. The important issue is the logic you used to arrive at your answer.
2. Use extra paper to determine your solutions then neatly transcribe them onto these sheets.
3. Do not submit the scratch sheets. However, all of the logic necessary to obtain the solution should be on these sheets.
4. Comment on all logical flaws and omissions and enclose

$\boxed{\textbf{comments in boxes}}$

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1. [5] Use a truth table to determine for which truth values of $p, q,$ and r $(\sim p \vee q) \wedge (r \Leftrightarrow p)$ is true.

2. [15] Using sentential calculus (with a four column format), prove that the conclusion $(s \wedge \sim p) \Rightarrow t$ follows from premises: $\sim (q \wedge s)$ and $q \vee p$. (Hint: Employ Conditionalization i.e., "Rule C".)

3. [15] Prove that the conclusion $p \Rightarrow s$ follows from the premises $\sim (p \wedge q), p \Rightarrow (q \vee r), \text{and } r \Rightarrow \sim p$. First convert the premises and the negation of the conclusion into Conjunctive Normal Form, and then employ a resolution proof to get a contradiction.

4. [15] Using the predicates defined on the set $\mathbb{N}$ of natural numbers:

Sxy x is a the successor of y (i.e. $x = y + 1$),

Exy x equal to y

Express in the syntax of Predicate Calculus:

a. *No natural number is a successor of itself*

$(\forall x)(\sim Sxx)$

b. *Every natural number has one and only one successor.*

$(\forall y)((\exists x)(Sxy \wedge (\forall z)(Szy \Rightarrow Exz)))$

c. *b is the successor of the successor of a.*

$(\exists x)(Sxa \wedge Sbx)$

5. [20] Prove that $(\exists z)Lz$ follows from $(\forall y)(\exists x)((Lx \Rightarrow Nx) \Rightarrow Gy)$ and $(\exists x)(\sim Gx)$.

6a. [10] Using induction, prove that for $n \geq 0$, $\sum_{k=0}^{n} (2k+1) = (n+1)^2$.

7. [10] Using induction, prove that for any real number a and for all integers $n, m \geq 1$, $a^{mn} = (a^m)^n$. You may assume for any real numbers α and β:

a. $\alpha^1 = \alpha$,
b. $\alpha^i \alpha^j = \alpha^{i+j}$, for all integers $i, j \geq 1$,
c. $\alpha^i \beta^i = (\alpha\beta)^i$, for all integers $i \geq 1$.

(Hint: Fix $n \geq 1$.)

8. [10] Prove for any sets A, B, C, and D that $(A \cap B) \sim (C \cap D) \subseteq (A \sim C) \cup (B \sim D)$.

9. [10]. Given sets $A, B,$ and C. Prove that $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

10. Let R be a relation on a non-empty set A. Define $R^1 = R$ and $R^{n+1} = R \circ R^n$ for $n \geq 1$.

a. [5]. Prove (from the definition) that if R is transitive $R^2 \subseteq R$.

b. [10]. Prove that if R is transitive $R^n \subseteq R$ for all $n \geq 1$. (You may assume that if $S_1 \subseteq S_2$ and $T_1 \subseteq T_2$ then $S_1 \circ T_1 \subseteq S_2 \circ T_2$.)

11. a.[10] Given a function $f: A \to B$, let E be a relation on A defined by $(x, y) \in E$ if and only if $f(x) = f(y)$. Prove that E is an equivalence relation.

b. (5) Let $A = \{-10, \ldots -1, 0, 1, 2, \ldots, 10\}$, $B = \mathbb{N}$ and $f(x) = x^2$. Specify the elements of the partition that E imposes on A. (Hint: Recall elements of the partition are sets.)

12. Given a non-empty set A and function $f: A \to A$,

a. [10] Prove that if $f \circ f$ is one-to-one then f is one-to-one.

b. [10] Prove that if $f \circ f$ is onto then f is onto.

13. [15] For any $n \geq 1$, consider the set $B = \{1, 2, \ldots, 2n\}$. Prove that if $A \subseteq B$ and $|A| \geq n+1$ then there exist $a, b \in A$ so that $a + b = 2n+1$. (Hint consider a function

$$f(x) = \begin{cases} x & x \leq n \\ 2n+1-x & x \geq n+1 \end{cases}.)$$