Final Examination Solutions

CS 313H

1. [5] Use a truth table to determine for which truth values of p, q, and r $\sim r \wedge (q \Leftrightarrow (p \vee \sim r))$ is true.

p	q	r	$\sim r$	$p \vee \sim r$	$q \Leftrightarrow (p \vee \sim r)$	$\sim r \wedge (q \Leftrightarrow (p \vee \sim r))$
F	F	F	T	T	F	F
F	F	T	F	F	T	F
F	T	F	T	T	T	T
F	T	T	F	F	F	F
T	F	F	T	T	F	F
T	F	T	F	T	F	F
T	T	F	T	T	T	T
T	T	T	F	T	T	F

2. [15] Using the rules of Sentential Calculus conclude $(p \wedge \sim b) \Rightarrow c$ from the premises $a \vee (b \vee c)$ and $p \Rightarrow \sim a$.

$\{P_1\}$	(1.) $a \vee (b \vee c)$	P
$\{P_2\}$	(2.) $p \Rightarrow \sim a$	P
$\{P_3\}$	(3.) $p \wedge \sim b$	P
$\{P_3\}$	(4.) p	Simp (3)
$\{P_2, P_3\}$	(5.) $\sim a$	MP (2), (4)
$\{P_3\}$	(6.) $\sim b$	Simp (3)
$\{P_2, P_3\}$	(7.) $\sim a \wedge \sim b$	Conj (5), (6)
$\{P_2, P_3\}$	(8.) $\sim (a \vee b)$	DeM (7)
$\{P_1\}$	(9.) $(a \vee b) \vee c$	Assoc (1)
$\{P_1, P_2, P_3\}$	(10.) c	DS (9)
$\{P_1, P_2\}$	(11.) $(p \wedge \sim b) \Rightarrow c$	C (3), (10)

3. [15] Prove that the conclusion p follows from premises: $p \vee q, q \Rightarrow t, \sim r \vee \sim s$, $(s \wedge t) \Rightarrow r$, and $q \Rightarrow s$. First convert the premises and the negation of the conclusion into Conjunctive Normal Form, and then employ a resolution proof to get a contradiction.

$p \vee q$

$q \Rightarrow t$

$\sim q \vee t$

$\sim r \vee \sim s$

$(s \wedge t) \Rightarrow r$

$\sim (s \wedge t) \vee r$

$\sim s \vee \sim t \vee r$

$q \Rightarrow s$

$\sim q \vee s$

$\sim p$

1. $p \vee q$	P
2. $\sim q \vee t$	P
3. $\sim r \vee \sim s$	P
4. $\sim s \vee \sim t \vee r$	P
5. $\sim q \vee s$	P
6. $\sim p$	P
7. q	Res (1), (6)
8. t	Res (2), (7)
9. $\sim s \vee r$	Res (4), (8)
10. $\sim s$	Res (3), (9)
11. s	Res (5), (7)
12. *false*	Conj. (10), (11)

4. [15] Using the predicates defined on the set P of people:

Lxy x likes y ,

Exy x equal to y

Gxy x goes with y ,

And the constants Alice, Bill, and Clyde,

Express in the syntax of Predicate Calculus:

a. *If anyone likes Alice it must be either Bill or Clyde.*

$$(\forall x)(LxAlice \Rightarrow (ExBill \vee ExClyde))$$

b. *Bill is the only one who will go with Clyde but no one will go with Alice.*

$$(GBillClyde \wedge (\forall x)(GxClyde \Rightarrow ExBill)) \wedge \sim (\exists y)(GyAlice)$$

c. *A person goes with another person only if the first one likes the second*

$$(\forall x)(\forall y)(Gxy \Rightarrow Lxy)$$

5. [20] Prove that $(\forall x)(Px \wedge Rx) \wedge (\exists y)Qy$ follows from $(\forall x)(Px \wedge (Qx \wedge Rx))$.

$\{P_1\}$	(1). $(\forall x)(Px \wedge (Qx \wedge Rx))$		P
$\{P_1\}$	(2). $Pa \wedge (Qa \wedge Ra)$		UI (1)
$\{P_1\}$	(3). $Pa \wedge (Ra \wedge Qa)$		Comm (2)
$\{P_1\}$	(4). $(Pa \wedge Ra) \wedge Qa$		Assoc (3)
$\{P_1\}$	(5). $Pa \wedge Ra$		Simp (4)
$\{P_1\}$	(6). $(\forall x)(Px \wedge Rx)$		UG (3)
$\{P_1\}$	(7). Qa		Simp (4)
$\{P_1\}$	(8). $(\exists y)Qy$		EG (3)
$\{P_1\}$	(9). $(\forall x)(Px \wedge Rx) \wedge (\exists y)Qy$		Conj (6)

6. [10] Using induction, prove that for $n \geq 0$, $3^{2n} + 4^{n+1}$ is an integer multiple of 5. (Hint: 9=5+4.)

For $n \geq 0$, let $P(n)$= "$3^{2n} + 4^{n+1}$ is an integer multiple of 5".
Basis step: $P(0)$ is true since $3^{2\cdot 0} + 4^{0+1} = 1 + 4 = 5 = 5 \cdot 1$..
Inductive step: For $n \geq 0$, $P(n) \Rightarrow P(n+1)$, since if $3^{2n} + 4^{n+1} = 5k$ for some interger k, then

$$\begin{aligned}3^{2(n+1)} + 4^{n+1+1} &= 9 \cdot 3^{2n} + 4 \cdot 4^{n+1}\\ &= (5+4) \cdot 3^{2n} + 4 \cdot 4^{n+1}\\ &= 5 \cdot 3^{2n} + 4(3^{2n} + 4^{n+1})\\ &= 5 \cdot 3^{2n} + 4 \cdot 5k\\ &= 5(3^{2n} + 4k)\end{aligned}$$

and $3^{2n} + 4k$ is an integer since $n \geq 0$ and k is an integer.

7. Prove or give a simple counterexample:

a. [5] For any sets $A, B,$ and C that $(A \sim B) \cap C = (A \cap C) \sim B$..

We have

$$\begin{aligned}& x \in (A \sim B) \cap C\\ &\Leftrightarrow (x \in A \sim B) \wedge (x \in C)\\ &\Leftrightarrow (x \in A \wedge) \wedge (x \in C)\\ &\Leftrightarrow (x \in A \wedge x \in C) \wedge (x \notin B)\\ &\Leftrightarrow x \in (A \cap C) \sim B.\end{aligned}$$

b. [5] For any sets $A, B,$ and C that $A \sim (B \cup C) = (A \sim B) \cup (A \sim C)$..

Let $A = B = \{1\}$ and $C = \varnothing$, then $A \sim (B \cup C) = \{1\} \sim (\{1\} \cup \varnothing) = \varnothing$ but $(A \sim B) \cup (A \sim C) = (\{1\} \sim \{1\}) \cup (\{1\} \sim \varnothing) = \varnothing \cup \{1\} = \{1\}$.

8. [10]. Given sets $A, B, C,$ and D be sets. Prove that $A \times B \subseteq C \times D$ if and only if $A \subseteq C \wedge B \subseteq D$.

Suppose $A \subseteq C \wedge B \subseteq D$, then
$(x, y) \in A \times B$
$\Rightarrow x \in A \wedge y \in B$
$\Rightarrow x \in C \wedge y \in D$
$\Rightarrow (x, y) \in C \times D$.

Suppose $A \times B \subseteq C \times D$, then
$x \in A \wedge y \in B$
$\Rightarrow (x, y) \in A \times B$
$\Rightarrow (x, y) \in C \times D$
$\Rightarrow x \in C \wedge y \in D$,
Thus $A \subseteq C \wedge B \subseteq D$.

10. [10} Prove that composition of relations is associative. That is if $R \subseteq A \times B, S \subseteq B \times C,$ and $T \subseteq C \times D,$ then $T \circ (S \circ R) = (T \circ S) \circ R$.

For $(a, d) \in A \times D$, we have
$(a, d) \in T \circ (S \circ R)$
$\Leftrightarrow \exists c \in C \ni (a, c) \in (S \circ R) \wedge (c, d) \in T$
$\Leftrightarrow (\exists c \in C \wedge \exists b \in B) \ni (a, b) \in R \wedge (b, c) \in S \wedge (c, d) \in T$
$\Leftrightarrow \exists b \in B \ni (a, b) \in R \wedge (b, d) \in T \circ S$
$\Leftrightarrow (a, d) \in (T \circ S) \circ R$.

11. [10] Let E be a relation on $\mathbb{R}^3$ (Cartesian three space) defined by $((x, y, z), (u, v, w)) \in E$ if and only if $x^2 + y^2 + z^2 = u^2 + v^2 + w^2$. Prove that E is an equivalence relation.

We must prove that E is reflexive, symmetric, and transitive. Since for all $(x, y, z) \in \mathbb{R}^3$, $x^2 + y^2 + z^2 = x^2 + y^2 + z^2$, we have $((x, y, z), (x, y, z)) \in E$ so E is reflexive. Next, since if $((x, y, z), (u, v, w)) \in E$ then $x^2 + y^2 + z^2 = u^2 + v^2 + w^2$ and $u^2 + v^2 + w^2 = x^2 + y^2 + z^2$ so $((u, v, w), (x, y, z)) \in E$ and E is symmetric. Finally, if $((x, y, z), (u, v, w)) \in E$ and $((u, v, w), (p, q, r)) \in E$ then $x^2 + y^2 + z^2 = u^2 + v^2 + w^2$ and $u^2 + v^2 + w^2 = p^2 + q^2 + r^2$ so $x^2 + y^2 + z^2 = p^2 + q^2 + r^2$ and $((x, y, z), (p, q, r)) \in E$, so E is transitive. Since E is reflexive, symmetric, and transitive it is an equivalence relation.

12. [10] Given sets A and B and functions $f : A \to B$, and $g : B \to A$. Prove that f and g are both one-to-one and onto then $g \circ f$ is one-to-one and onto.

> Since f is one-to-one if $x \neq y$ then $f(x) \neq f(y)$. But then since g is one-to-one $g \circ f(x) g(f(x)) \neq g(f(y)) = g \circ f(y)$, so $g \circ f$ is one-to-one. Given $x \in A$, since g is onto, there exists a $y \in B$ so that $g(y) = x$. But then since f is onto, there exists a $z \in A$ so that $f(z) = y$, but then for that $z \in A$ $g \circ f(x) = g(f(x)) = g(y) = x$, so $g \circ f$ is onto.

13. [10] Consider the parity function on integers: $p : \mathbb{Z} \to \{0,1\}$ where $p(n) = \begin{cases} 0 & n \, is \, even \\ 1 & n \, is \, odd \end{cases}$ and extend it to triples of integers by $p_3 : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \to \{0,1\} \times \{0,1\} \times \{0,1\}$ where $p_3(m,n,r) = (p(m), p(n), p(r))$. Show that given any nine points in $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, two must have the same parity.

> Let A represent the set of nine triples of integers. Since $\{0,1\} \times \{0,1\} \times \{0,1\}$ has only eight elements, the function p_3 cannot be one-to-one on A. Thus there must exist two triples in the set with the same parity.