General Comments on Proofs

These comments follow my portion of the grading of the second examination, but I have seen similar problems on other exams and with other classes.

1. A Proof Need Not Display How it Was Discovered. This is a problem I see so often. Somehow students believe that they must show how they discovered the proof. First: *No, you don't.* A proof is valid based on its argument. One can judge the correctness or incorrectness of an argument without having any idea of how it was discovered. Second: *It might be interesting to know how it was discovered, but including that in the proof not only adds to the length but may diminish the clarity.* Arguments already might be quite complex – adding irrelevancies may increase the likelihood of confusion. Third: *With students just beginning to do formal proofs, there is a danger that a display of a discovery actually substitutes for a proof.* Thus, students will present arguments that go through some large numbers of transformations that end in something true, but may or may not be reversible. They may have discovered something that can be turned into a proof but the discovery process itself is not a proof.

2. If given an equation to prove, don't present an argument that assumes it is correct and manipulates both sides. Consider this gross example:

Prove: $\sqrt{\pi} = -\sqrt{3\pi}$

Proof:

$$\begin{aligned} \sqrt{\pi} &= -\sqrt{3\pi} \\ (\sqrt{\pi})^2 &= (-\sqrt{3\pi})^2 \\ \pi &= 3\pi \\ \sin(\pi) &= \sin(3\pi) \\ 0 &= 0. QED \end{aligned}$$

The bad idea is to assume the equality and then manipulate both sides until equality is obvious. The trouble is that the steps may not be reversible. In the case above there are actually two steps (the squaring and the sine) that are not reversible.

In response to this complaint students sometimes say "But my steps are reversible.". If that is true then it is easy to transform into a proper argument. To be precise, suppose you have something of the form:

Prove: $\exp 1 = \exp 2$

Proof:

$$\begin{aligned} \exp 1 &= \exp 2 \\ \exp 3 &= \exp 4 \\ &\vdots \\ \exp 17 &= \exp 18 \end{aligned}$$

that you claim is reversible. Fine. If it is reversible then present it in this fashion:

$$\begin{aligned} \exp 1 &= \exp 3 \\ &= \exp 5 \\ &\vdots \\ &= \exp 17 \\ &= \exp 18 \\ &= \exp 16 \\ &\vdots \\ &= \exp 2 \end{aligned}$$

The expressions here can be logical statements, arithmetic, sets, whatever. The point is that you have no right to insist that the reader of your proof is **1.** to guess that you mean that the proof actually is be read down the left column and up the right column and **2.** to determine which steps are reversible.

This interacts with Comment 1 above. The manipulation of both sides of an equation is sometimes how a person **discovers** a proof. In a desire to display this discovery (which is unnecessary) they forget to display the proper steps.

3. All statements must be readable. There is a temptation when using mathematical notation to forget that these statements must have subjects and verbs just like any statement. Also, students new to notations may mix things in a fashion that makes no sense. Try to read this:

$$\begin{aligned} &\Leftrightarrow p_1 \\ &\Leftrightarrow p_2 \\ &\vdots \\ &\Leftrightarrow p_{17} \end{aligned}$$

Did you hear yourself begin with "If and only if p_1 if and only if p_2 …"? What in the world does that mean? Had the person writing it actually read it, he or she would have said "I meant 'p_1 if and only if p_2 …' ". Fine. Then it should have been written: like this:

$$\begin{aligned} &\quad p_1 \\ &\Leftrightarrow p_2 \\ &\vdots \\ &\Leftrightarrow p_{17} \end{aligned}$$

Similarly, try to red this:

$$\begin{aligned} &\text{Let } x \in A \\ &\Leftrightarrow x \in B \\ &\vdots \end{aligned}$$

Should that really be read "Let x be an element of A if and only if x is an element of B…"? I think what is meant is "x is an element of A if and only if x is an element of B…"? and should be written without the "Let".

Next try to read these:

$$(A \vee B) \sim C$$
$$(A \vee B) \cap C$$
$$P(n) = (A \cap B_n)$$
$$(A \cup B) \sim C = ((x \in A \vee x \in B) \wedge x \notin C)$$

The first and second make no sense. They mix logical notation and set theoretic notation. The third is fine **IF** *P(n)* is a set. It makes no sense if *P(n)* is an assertion since the left side is an assertion and the right side is a set. The last example is the same except that the set is on the left and the assertion is on the right.

4. Lines in arguments are assumed to follow from previous lines – not to be equivalent to previous lines unless that is explicitly said. Here's a good example:

$$x \in A_1$$
$$x \in A_2$$
$$\vdots$$
$$x \in A_{17}$$

is an argument that $A_1 \subseteq A_{17}$. Did you think it was an argument that $A_1 = A_{17}$? Our assumption is that lines follow from previous lines – **NOT** that they are equivalent. If you mean to say they are equivalent that must be explicit:

$$x \in A_1$$
$$\Leftrightarrow x \in A_2$$
$$\vdots$$
$$\Leftrightarrow x \in A_{17}$$

The same thing with arithmetic equalities (but that was said up in Comment 2).

You can eliminate a large number of these problems simply by verbalizing what you have written. If you can't say it without making nonsense, you don't want it.