1. Using induction prove for $n \geq 2$, that $\prod_{k=2}^{n}(1-\frac{1}{k^2}) = \frac{n+1}{2n}$.

For $n = 2$, we have

$$\prod_{k=2}^{n}(1-\frac{1}{k^2}) = \prod_{k=2}^{2}(1-\frac{1}{k^2})$$
$$= 1-\frac{1}{4}$$
$$= \frac{3}{4}$$
$$= \frac{2+1}{2\cdot 2}$$
$$= \frac{n+1}{2n}.$$

Now assume for some $n \geq 2$, that $\prod_{k=2}^{n}(1-\frac{1}{k^2}) = \frac{n+1}{2n}$. We then have

$$\prod_{k=2}^{n+1}(1-\frac{1}{k^2}) = \prod_{k=2}^{n}(1-\frac{1}{k^2})(1-\frac{1}{(n+1)^2})$$
$$= \frac{n+1}{2n}(1-\frac{1}{(n+1)^2})$$
$$= \frac{n+1}{2n}\frac{(n+1)^2-1}{(n+1)^2}$$
$$= \frac{n+1}{2n}\frac{n(n+2)}{(n+1)^2}$$
$$= \frac{n+2}{2(n+1)}$$
$$= \frac{(n+1)+1}{2(n+1)}.$$

2. Using induction, prove that for $n \geq 0$, $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$

For $n = 1$, we have $\sum_{k=0}^{0} 2^k = 2^0 = 1 = 2^{0+1} - 1$. Now assume the result is true for some $n \geq 0$. We then have $\sum_{k=0}^{n+1} 2^k = \sum_{k=0}^{n} 2^k + 2^{n+1} = (2^{n+1} - 1) + 2^{n+1} = 2 \cdot 2^{n+1} - 1 = 2^{(n+1)+1} - 1$. The result then holds for all $n \geq 0$.

3. Assuming $l \neq 0$ and $l \neq 1$ and using induction, prove that for $n \geq 0$, $\sum_{k=0}^{n} l^k = \frac{l^{n+1}-1}{l-1}$.

For $n=0$, we have $\sum_{k=0}^{0} l^k = l^0 = 1 = \frac{l^{0+1}-1}{l-1}$. Now assume the result is true for $n \geq 0$. We then have $\sum_{k=0}^{n+1} l^k = \sum_{k=0}^{n} l^k + l^{n+1} = \frac{l^{n+1}-1}{l-1} + l^{n+1} = \frac{l^{n+1}-1+l^{n+2}-l^{n+1}}{l-1} = \frac{l^{n+2}-1}{l-1}$. The result then holds for all $n \geq 0$.

4. Using induction, prove that for $n \geq 1$, n^3+2n is an integral multiple of 3 (i.e. $\forall n \in \mathbb{Z}((n \geq 1) \Rightarrow \exists k \in \mathbb{Z}(n^3+2n = 3k))$.

For $n=1$, we have $n^3+2n=3=3\cdot 1$. Now assume the result is true for $n \geq 1$. We then have some integer k so that have $n^3+2n=3k$. But then

$$\begin{aligned}(n+1)^3+2(n+1) &= n^3+3n^2+3n+1+2n+2\\ &= n^3+2n+3(n^2+n+1)\\ &= 3k+3(n^2+n+1)\\ &= 3\cdot(k+n^2+n+1).\end{aligned}$$

Since n is an integer, so is $k+n^2+n+1$ so $(n+1)^3+2(n+1)$ is an integral multiple of 3. The result then holds for all $n \geq 1$.

5. Using induction, prove that for $n \geq 1$, $\sum_{k=1}^{n}(4k-3) = n(2n-1)$.

For $n=1$, we have $\sum_{k=1}^{n}(4k-3) = 4-3 = 1 = 1(2\cdot 1-1) = n(2n-1)$. Now assume the result is true for $n \geq 1$. We then have

$$\begin{aligned}\sum_{k=1}^{n+1}(4k-3) &= \sum_{k=1}^{n}(4k-3)+(4(n+1)-3)\\ &= n(2n-1)+4n+4-3\\ &= 2n^2-n+4n+1\\ &= 2(n+1)^2-(n+1)\\ &= (n+1)(2(n+1)-1).\end{aligned}$$

The result then holds for all $n \geq 1$.

6. For fixed real numbers a and b, consider the iteratively defined sequence:

$$s_0 = a$$
$$s_n = 2s_{n-1} + b, \text{ for } n \geq 1.$$

Using induction, prove that for $n \geq 0, s_n = 2^n a + (2^n - 1)b$.

For $n = 0$, we have $s_0 = a = 1 \cdot a + (1-1)b = 2^0 a + (2^0 - 1)b$. Now assume the result is true for some $n \geq 0$. We then have

$$\begin{aligned} s_{n+1} &= 2s_n + b \\ &= 2(2^n a + (2^n - 1)b) + b \\ &= 2^{n+1} a + (2^{n+1} - 2 + 1)b \\ &= 2^{n+1} a + (2^{n+1} - 1)b. \end{aligned}$$

The result then holds for $n+1$ and by induction holds for all $n \geq 0$.

7. Using induction, prove that for $n \geq 4$, $n! > 2^n$.

For $n = 4$, we have $n! = 4! = 24 > 16 = 2^4 = 2^n$. Now assume the result is true for some $n \geq 4$. We then have $n+1 \geq 5 > 2$ so

$$(n+1)! = (n+1) \cdot n! > (n+1) \cdot 2^n > 2 \cdot 2^n = 2^{n+1}.$$

The result then holds for $n+1$ and by induction holds for all $n \geq 4$.

8. For fixed real numbers a and b, with $a \neq 1$, define

$$x_0 = 0,$$

and for $k = 1, 2, \ldots$

$$x_k = a\, x_{k-1} + b.$$

Using induction, prove that for $k \geq 0$, $x_k = \dfrac{a^k - 1}{a - 1} b$.

Consider the inductive hypothesis, $P(k) = "x_k = \dfrac{a^k - 1}{a - 1} b"$. For $k = 0$, we have $x_0 = 0 = \dfrac{a^0 - 1}{a - 1} b$. So $P(0)$ is true. Now for some $k \geq 1$, assume $P(k)$. We then have

$$\begin{aligned} x_{k+1} &= a\, x_k + b \\ &= a \frac{a^k - 1}{a - 1} b + b \\ &= \frac{a^{k+1} - a + a - 1}{a - 1} b \\ &= \frac{a^{k+1} - 1}{a - 1} b, \end{aligned}$$

so $P(k+1)$ is also true. By induction we have for $k \geq 0$, $x_k = \dfrac{a^k - 1}{a - 1} b$.

9. Consider the Fibonacci sequence: $f_0 = 1,\ f_1 = 1,\ f_k = f_{k-1} + f_{k-2}$, for $k \geq 2$. Using induction, prove that for $n \geq 0$, $\sum_{k=0}^{n} f_k^2 = f_n f_{n+1}$.

For $n \geq 0$, let $P(n)$ = "$\sum_{k=0}^{n} f_k^2 = f_n f_{n+1}$".

Basis step: $P(0)$ is true since $\sum_{k=0}^{n} f_k^2 = f_0^2 = 1 = 1 \cdot 1 = f_0 f_1$..

Inductive step: For $n \geq 0$, $P(n) \Rightarrow P(n+1)$, since if $\sum_{k=0}^{n} f_k^2 = f_n f_{n+1}$, then

$$\begin{aligned}\sum_{k=0}^{n+1} f_k^2 &= \sum_{k=0}^{n} f_k^2 + f_{n+1}^2 \\ &= f_n f_{n+1} + f_{n+1}^2 \\ &= f_{n+1}(f_n + f_{n+1}) \\ &= f_{n+1} f_{(n+1)+1}.\end{aligned}$$

10. Using induction, prove that for $n \geq 0$, $\sum_{k=0}^{n} (2k+1) = (n+1)^2$.

For $n \geq 0$, let $P(n)$ = "$\sum_{k=0}^{n} (2k+1) = (n+1)^2$".

Basis step: $P(0)$ is true since $\sum_{k=0}^{0} (2k+1) = (2 \cdot 0 + 1) = 1 = (0+1)^2$.

Inductive step: For $n \geq 0$, $P(n) \Rightarrow P(n+1)$, since if $\sum_{k=0}^{n} (2k+1) = (n+1)^2$, then

$$\begin{aligned}\sum_{k=0}^{n+1} (2k+1) &= \sum_{k=0}^{n} (2k+1) + 2(n+1) + 1 \\ &= (n+1)^2 + 2(n+1) + 1 \\ &= ((n+1)+1)^2.\end{aligned}$$

11. Using induction, prove that for any real number a and for all integers $n, m \geq 1$, $a^{mn} = (a^m)^n$. You may assume for any real numbers a and b:

a. $a^1 = a$,
b. $a^i a^j = a^{i+j}$, for all integers $i, j \geq 1$,
c. $a^i b^i = (ab)^i$, for all integers $i \geq 1$.

(Hint: Fix $n \geq 1$.)

Fix $n \geq 1$. For $m \geq 1$, let $P(m)$ = "$a^{mn} = (a^m)^n$".
Basis step: $P(1)$ is true since $a^{1 \cdot n} = a^n = (a^1)^n$.
Inductive step on m: For $m \geq 1$, $P(m) \Rightarrow P(m+1)$, since if $a^{mn} = (a^m)^n$ then

$$\begin{aligned} a^{(m+1)n} &= a^{mn+n} = a^{mn}a^n \\ &= (a^m)^n a^n \\ &= (a^m a)^n \\ &= (a^{m+1})^n. \end{aligned}$$

12. Using induction, prove that for $n \geq 2$, $2n + 3 < n^3$.

For $n \geq 2$, let $P(n)$ = "$2n + 3 < n^3$".
Basis step: $P(2)$ is true since $2 \cdot 2 + 3 = 7 < 8 = 2^3$.
Inductive step: For $n \geq 2$, $P(n) \Rightarrow P(n+1)$, since if $2n + 3 < n^3$, then $3n > 2$ and $3n^2 + 1 > 0$ so:

$$\begin{aligned} 2(n+1) + 3 &= 2n + 3 + 2 \\ &< n^3 + 2 \\ &< n^3 + 3n \\ &< n^3 + 3n^2 + 3n + 1 = (n+1)^3. \end{aligned}$$

13. Using induction, prove that for $n \geq 0$, $1 \leq 3^n$.

For $n \geq 0$, let $P(n)$ = "$1 \leq 3^n$".
Basis step: $P(0)$ is true since $1 \leq 1 = 3^0$. .
Inductive step: For $n \geq 0$, $P(n) \Rightarrow P(n+1)$, since if $1 \leq 3^n$, then

$$\begin{aligned} 1 &\leq 3 \\ &\leq 3 \cdot 3^n \\ &\leq 3^{n+1}. \end{aligned}$$

14. Using induction and Problem 13, prove that for $n \geq 2$, $1+2n < 3^n$.

For $n \geq 2$, let $P(n)$ = "$1+2n < 3^n$".
Basis step: $P(2)$ is true since $1+2 \cdot 2 = 5 < 9 = 3^2$..
Inductive step: For $n \geq 0$, $P(n) \Rightarrow P(n+1)$, since if $1+2n < 3^n$, then

$$\begin{aligned} 1+2(n+1) &= 1+2n+2 \\ &< 3^n + 2 \\ &< 3^n + 2 \cdot 1 \\ &< 3^n + 2 \cdot 3^n \\ &< 3^{n+1}. \end{aligned}$$

15. Consider the sequence: $a_0 = 2, a_1 = 1, a_n = a_{n-1} + 2a_{n-2}$, for $n \geq 2$. Using induction, prove that for $n \geq 0$, $a_n = 2^n + (-1)^n$.

For $n \geq 0$, let $P(n)$ = "$a_n = 2^n + (-1)^n$".
Basis step: $P(0)$ and $P(1)$ are true since $a_0 = 2 = 1+1 = 2^0 + (-1)^0$ and $a_1 = 1 = 2-1 = 2^1 + (-1)^1$.
Inductive step: For $n \geq 2$, $P(n) \Rightarrow P(n+1)$, since if $a_{n-2} = 2^{n-2} + (-1)^{n-2}$ and $a_{n-1} = 2^{n-1} + (-1)^{n-1}$, then

$$\begin{aligned} a_n &= a_{n-1} + 2a_{n-2} \\ &= 2^{n-1} + (-1)^{n-1} + 2(2^{n-2} + (-1)^{n-2}) \\ &= 2^{n-1} - (-1)^n + 2^{n-1} + 2(-1)^n \\ &= 2 \cdot 2^{n-1} + (-1)^n \\ &= 2^n + (-1)^n. \end{aligned}$$