Large Scale Distributivity

We have two little distributivity tautologies:

$$(j \vee (y \wedge c)) \Leftrightarrow ((j \vee y) \wedge (j \vee c))$$

and

$$(j \wedge (y \vee c)) \Leftrightarrow ((j \wedge y) \vee (j \wedge c)).$$

What happens when we have more than three sentences?

We simply need to use the rules several times.

Let's deal with the "or-over-and" case from above first. In particular, this applies in the reduction to CNF if, on the way to a conjunction of disjunctive clauses, you actually have a disjunction of conjunctive clauses (that we could call Disjunctive Normal Form). So we get

$$\begin{aligned}((j \wedge q) \vee (y \wedge c)) &\Leftrightarrow (((j \wedge q) \vee y) \wedge ((j \wedge q) \vee c)) \\ &\Leftrightarrow (((j \vee y) \wedge (q \vee y)) \wedge ((j \vee c) \wedge (q \vee c))) \\ &\Leftrightarrow (j \vee y) \wedge (q \vee y) \wedge (j \vee c) \wedge (q \vee c)\end{aligned}$$

(Notice in the final line I shamelessly removed the parenthesis to make it clearer.)

Notice what we have in the end: a conjunction of every disjunction of an element of $\{j, q\}$ with an element of $\{y, c\}$. So what do you think would result with even more – for example $(j_1 \wedge j_2 \wedge ... \wedge j_m) \vee (y_1 \wedge y_2 \wedge ... \wedge y_n)$? To help you remember these things, you might want to think of an analogous arithmetic identity. You know that

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

And you also know that

$$(a + d) \cdot (b + c) = (a \cdot b) + (a \cdot c) + (d \cdot b) + (d \cdot c).$$

Notice you have the sum of every product of an element of $\{a, d\}$ with an element of $\{b, c\}$. (Recall in the logical version you had a conjunction of every disjunction of an element of $\{j, q\}$ with an element of $\{y, c\}$.) That ought to be a memory clue and also a major suggest of what you should get for the large version $(j_1 \wedge j_2 \wedge ... \wedge j_m) \vee (y_1 \wedge y_2 \wedge ... \wedge y_n)$.

So now push yourself:

1. What is the CNF of $(j_1 \wedge j_2 \wedge ... \wedge j_m) \vee (y_1 \wedge y_2 \wedge ... \wedge y_n)$?

$$\begin{aligned}&(j_1 \vee y_1) \wedge (j_1 \vee y_2) \wedge ... \wedge (j_1 \vee y_n) \\ &\wedge (j_2 \vee y_1) \wedge (j_2 \vee y_2) \wedge ... \wedge (j_2 \vee y_n) \\ &\wedge ... \wedge (j_m \vee y_1) \wedge (j_m \vee y_2) \wedge ... \wedge (j_m \vee y_n)\end{aligned}$$

2. What should you get if you reverse the conjunctions and disjunctions? That is, what is equivalent to $(j \vee q) \wedge (y \vee c)$? (Does the arithmetic analog still hold?)

$$
\begin{aligned}
((j \vee q) \wedge (y \vee c)) &\Leftrightarrow (((j \vee q) \wedge y) \vee ((j \vee q) \wedge c)) \\
&\Leftrightarrow (((j \wedge y) \vee (q \wedge y)) \vee ((j \wedge c) \vee (q \wedge c))) \\
&\Leftrightarrow (j \wedge y) \vee (q \wedge y) \vee (j \wedge c) \vee (q \wedge c)
\end{aligned}
$$

The arithmetic analog holds but now addition is analogous to disjunction and multiplication is analogous to conjunction.

3. What is the DNF of $(j_1 \vee j_2 \vee \ldots \vee j_m) \wedge (y_1 \vee y_2 \vee \ldots \vee y_n)$?

$$
\begin{aligned}
&(j_1 \wedge y_1) \vee (j_1 \wedge y_2) \vee \ldots \vee (j_1 \wedge y_n) \\
&\vee (j_2 \wedge y_1) \wedge (j_2 \wedge y_2) \vee \ldots \vee (j_2 \wedge y_n) \\
&\vee \ldots \vee (j_m \wedge y_1) \vee (j_m \wedge y_2) \vee \ldots \vee (j_m \wedge y_n)
\end{aligned}
$$

4. What is the CNF of $(j_1^1 \wedge j_2^1 \wedge \ldots \wedge j_{m_1}^1) \vee (j_1^2 \wedge j_2^2 \wedge \ldots \wedge j_{m_2}^2) \vee \ldots \vee (j_1^n \wedge j_2^n \wedge \ldots \wedge j_{m_n}^n)$?

It is the conjunction of $m_1 m_2 \cdots m_n$ disjunctive clauses. Each of those clauses has the disjunction of one element of $\{j_1^1, j_2^1, \ldots, j_{m_1}^1\}$, one element of $\{j_1^2, j_2^2, \ldots, j_{m_2}^2\}$, …., and one element of $\{j_1^n, j_2^n, \ldots, j_{m_n}^n\}$.

5. What is the DNF of $(j_1^1 \vee j_2^1 \vee \ldots \vee j_{m_1}^1) \wedge (j_1^2 \vee j_2^2 \vee \ldots \vee j_{m_2}^2) \wedge \ldots \wedge (j_1^n \vee j_2^n \vee \ldots \vee j_{m_n}^n)$?

It is the disjunction of $m_1 m_2 \cdots m_n$ conjunctive clauses. Each of those clauses has the conjunction of one element of $\{j_1^1, j_2^1, \ldots, j_{m_1}^1\}$, one element of $\{j_1^2, j_2^2, \ldots, j_{m_2}^2\}$, …., and one element of $\{j_1^n, j_2^n, \ldots, j_{m_n}^n\}$.