Large Scale Distributivity

We have two little distributivity tautologies:

\[(\phi \lor (\psi \land \chi)) \iff ((\phi \lor \psi) \land (\phi \lor \chi))\]

and

\[(\phi \land (\psi \lor \chi)) \iff ((\phi \land \psi) \lor (\phi \land \chi))\].

What happens when we have more than three sentences?

We simply need to use the rules several times.

Let's deal with the “or-over-and” case from above first. In particular, this applies in the reduction to CNF if, on the way to a conjunction of disjunctive clauses, you actually have a disjunction of conjunctive clauses (that we could call Disjunctive Normal Form). So we get

\[
((\phi \land \theta) \lor (\psi \land \chi)) \iff (((\phi \land \theta) \lor \psi) \land ((\phi \land \theta) \lor \chi))
\iff (((\phi \land \psi) \land (\theta \lor \psi)) \land ((\phi \land \chi) \land (\theta \lor \chi))
\iff (\phi \land \psi) \lor (\theta \lor \psi) \land (\phi \land \chi) \lor (\theta \lor \chi)
\]

(Notice in the final line I shamelessly removed the parenthesis to make it clearer.)

Notice what we have in the end: a conjunction of every disjunction of an element of \(\{\phi, \theta\}\) with an element of \(\{\psi, \chi\}\). So what do you think would result with even more – for example \((\phi_1 \land \phi_2 \land \ldots \land \phi_m) \lor (\psi_1 \land \psi_2 \land \ldots \land \psi_n)\)? To help you remember these things, you might want to think of an analogous arithmetic identity. You know that

\[a \cdot (b + c) = (a \cdot b) + (a \cdot c)\]

And you also know that

\[(a + d) \cdot (b + c) = (a \cdot b) + (a \cdot c) + (d \cdot b) + (d \cdot c)\].

Notice you have the sum of every product of an element of \(\{a,d\}\) with an element of \(\{b,c\}\). (Recall in the logical version you had a conjunction of every disjunction of an element of \(\{\phi, \theta\}\) with an element of \(\{\psi, \chi\}\).) That ought to be a memory clue and also a major suggestion of what you should get for the large version \((\phi_1 \land \phi_2 \land \ldots \land \phi_m) \lor (\psi_1 \land \psi_2 \land \ldots \land \psi_n)\).

So now push yourself:

1. What is the CNF of \((\phi_1 \land \phi_2 \land \ldots \land \phi_m) \lor (\psi_1 \land \psi_2 \land \ldots \land \psi_n)\)?

\[
(\phi_1 \lor \psi_1) \land (\phi_1 \lor \psi_2) \land \ldots \land (\phi_1 \lor \psi_n)
\land (\phi_2 \lor \psi_1) \land (\phi_2 \lor \psi_2) \land \ldots \land (\phi_2 \lor \psi_n)
\land \ldots \land (\phi_m \lor \psi_1) \land (\phi_m \lor \psi_2) \land \ldots \land (\phi_m \lor \psi_n)
\]
2. What should you get if you reverse the conjunctions and disjunctions? That is, what is equivalent to \((\phi \lor \theta) \land (\psi \lor \chi)\)? (Does the arithmetic analog still hold?)

\[
((\phi \lor \theta) \land (\psi \lor \chi)) \iff (((\phi \lor \theta) \land \psi) \lor ((\phi \lor \theta) \land \chi)) \\
\iff ((\phi \land \psi) \lor (\theta \land \psi)) \lor ((\phi \land \chi) \lor (\theta \land \chi)) \\
\iff (\phi \land \psi) \lor (\theta \land \psi) \lor (\phi \land \chi) \lor (\theta \land \chi)
\]

The arithmetic analog holds but now addition is analogous to disjunction and multiplication is analogous to conjunction.

3. What is the DNF of \((\phi_1 \lor \phi_2 \lor \ldots \lor \phi_m) \land (\psi_1 \lor \psi_2 \lor \ldots \lor \psi_n)\)?

\[
(\phi_1 \land \psi_1) \lor (\phi_1 \land \psi_2) \lor \ldots \lor (\phi_1 \land \psi_n) \\
(\phi_2 \land \psi_1) \lor (\phi_2 \land \psi_2) \lor \ldots \lor (\phi_2 \land \psi_n) \\
\ldots \\
(\phi_m \land \psi_1) \lor (\phi_m \land \psi_2) \lor \ldots \lor (\phi_m \land \psi_n)
\]

4. What is the CNF of \((\phi_1^1 \land \phi_2^1 \land \ldots \land \phi_{mn}^1) \lor (\phi_1^2 \land \phi_2^2 \land \ldots \land \phi_{mn}^2) \lor \ldots \lor (\phi_1^n \land \phi_2^n \land \ldots \land \phi_{mn}^n)\)?

It is the conjunction of \(m \cdot m_2 \cdots m_n\) disjunctive clauses. Each of those clauses has the disjunction of one element of \(\{\phi_1^1, \phi_2^1, \ldots, \phi_{mn}^1\}\), one element of \(\{\phi_1^2, \phi_2^2, \ldots, \phi_{mn}^2\}\), \ldots, and one element of \(\{\phi_1^n, \phi_2^n, \ldots, \phi_{mn}^n\}\).

5. What is the DNF of \((\phi_1^1 \lor \phi_2^1 \lor \ldots \lor \phi_{mn}^1) \land (\phi_1^2 \lor \phi_2^2 \lor \ldots \lor \phi_{mn}^2) \land \ldots \land (\phi_1^n \lor \phi_2^n \lor \ldots \lor \phi_{mn}^n)\)?

It is the disjunction of \(m \cdot m_2 \cdots m_n\) conjunctive clauses. Each of those clauses has the conjunction of one element of \(\{\phi_1^1, \phi_2^1, \ldots, \phi_{mn}^1\}\), one element of \(\{\phi_1^2, \phi_2^2, \ldots, \phi_{mn}^2\}\), \ldots, and one element of \(\{\phi_1^n, \phi_2^n, \ldots, \phi_{mn}^n\}\).