Review of Mathematical Concepts

1 Sets

1.1 What is a Set?

A ***set*** is simply a collection of objects. The objects (which we call the ***elements*** or ***members*** of the set) can be anything: numbers, people, fruits, whatever. For example, all of the following are sets:

A = {13, 11, 8, 23}

B = {8, 23, 11, 13}

C = {8, 8, 23, 23, 11, 11, 13, 13}

D = {apple, pear, banana, grape}

E = {January, February, March, April, May, June, July, August, September, October, November, December}

F = $\{x : x \in E$ and x has 31 days$\}$

G = {January, March, May, July, August, October, December}

N = the nonnegative integers (We will generally call this set N, the ***natural numbers***.)

H = $\{i : \exists x \in N$ and $i = 2x\}$

I = {0, 2, 4, 6, 8, ... }

J = the even natural numbers

K = the syntactically valid C programs

L = $\{x : x \in K$ and x never gets into an infinite loop$\}$

Z = the integers (... -3, -2, -1, 0, 1, 2, 3, ...)

In the definitions of F and H, we have used the colon notation. Read it as "such that". We've also used the standard symbol $\in$ for "element of". We will also use $\notin$ for "not an element of". So, for example, $17 \notin A$ is true.

Remember that a set is simply a collection of elements. So if two sets contain precisely the same elements (regardless of the way we actually defined the sets), then they are identical. Thus F and G are the same set, as are H, I, and J.

Since a set is defined only by what elements it contains, it does not matter what order we list the elements in. Thus A and B are the same set.

Our definition of a set considers only whether or not an element is contained within the set. It does not consider how many times the element is mentioned. In other words, duplicates don't count. So A, B, and C are all equal.

The smallest set is the set that contains no elements. It is called the ***empty set***, and is written $\varnothing$ or $\{\}$.

When you are working with sets, it is very important to keep in mind the difference between a set and the elements of a set. Given a set that contains more than one element, this not usually tricky. It's clear that $\{1, 2\}$ is distinct from either the number 1 or the number 2. It sometimes becomes a bit trickier though with ***singleton sets*** (sets that contain only a single element). But it is equally true here. So, for example, $\{1\}$ is distinct from the number 1. As another example, consider $\{\varnothing\}$. This is a set that contains one element. That element is in turn a set that contains no elements (i.e., the empty set).

1.2 Relating Sets to Each Other

We say that A is a ***subset*** of B (which we write as $A \subseteq B$) if every element of A is also an element of B. The symbol we use for subset ($\subseteq$) looks somewhat like $\leq$. This is no accident. If $A \subseteq B$, then there is a sense in which the set A is "less than or equal to" the set B, since all the elements of A must be in B, but there may be elements of B that are not in A.

Given this definition, notice that every set is a subset of itself. This fact turns out to offer us a useful way to prove that two sets A and B are equal: First prove that A is a subset of B. Then prove that B is a subset of A. We'll have more to say about this later in Section 6.2.

We say that A is ***proper subset*** of B (written $A \subset B$) if $A \subseteq B$ and $A \neq B$.

Notice that the empty set is a subset of every set (since, trivially, every element of $\varnothing$, all none of them, is also an element of every other set). And the empty set is a *proper* subset of every set other than itself.