Some examples of strong induction

Template: $P(n_0) \wedge (((n_0 \le i \le n) \Rightarrow P(i)) \Rightarrow P(n+1))$

1. Using strong induction, I will prove that every positive integer can be written as a sum of distinct powers of 2. Thus for $n \ge 1$, $P(n) =$ "n can be written as a sum of distinct powers of 2". $P(1)$ is true since $1 = 2^0$. Now consider any $n \ge 1$. There exists an integer k so that

$$2^k \le n+1 < 2^{k+1} .$$

If $2^k = n+1$, then $n+1$ can be written as a sum of distinct powers of 2. If $2^k \ne n+1$, then

$$2^k < n+1 < 2^{k+1}$$

so

$$0 < n+1-2^k < 2^{k+1} - 2^k = 2^k \le n .$$

Since the value of $n+1-2^k$ is positive but less than n, the inductive hypothesis guarantees that $n+1-2^k$ can be written as a sum of distinct powers of 2 and the powers are less than k. Thus $n+1 = 2^k +$ a sum of distinct powers of 2 and the powers are distinct.

2. Using strong induction, I will prove that the Fibonacci sequence:

$$a_0 = 1,$$
$$a_1 = 1,$$
$$a_{k+1} = a_k + a_{k+1}, \text{for } k \ge 1.$$

satisfies for $k \ge 1$,

$$a_k \ge \left(\frac{3}{2}\right)^{k-2} .$$

Thus for $k \ge 1$, $P(k) =$ "$a_k \ge \left(\frac{3}{2}\right)^{k-2}$". $P(1)$ is true since $a_1 = 1 \ge \frac{2}{3} = \left(\frac{3}{2}\right)^{1-2}$. Now consider any $k \ge 1$. If we assume $a_{k-1} \ge \left(\frac{3}{2}\right)^{k-3}$ and $a_k \ge \left(\frac{3}{2}\right)^{k-2}$, then

$$\begin{aligned}
a_{k+1} = a_k + a_{k-1} &\ge \left(\frac{3}{2}\right)^{k-2} + \left(\frac{3}{2}\right)^{k-3} \\
&\ge \left(\frac{3}{2}+1\right)\left(\frac{3}{2}\right)^{k-3} \\
&\ge \left(\frac{5}{2}\right)\left(\frac{3}{2}\right)^{k-3} \\
&\ge \left(\frac{9}{4}\right)\left(\frac{3}{2}\right)^{k-3} \\
&\ge \left(\frac{3}{2}\right)^{2}\left(\frac{3}{2}\right)^{k-3} = \left(\frac{3}{2}\right)^{(k+1)-2}
\end{aligned}$$

3. Using strong induction, I will prove that integer larger than one has a prime factor. Thus for $n \geq 2$, $P(n) =$ "n has a prime factor". $P(2)$ is true since the prime 2 divides 2. Now consider any $n \geq 2$. The integer $n+1$ is either prime or not. If it is prime then it has a prime factor. If $n+1$ is not prime then it has some factor k satisfying

$$2 \leq k < n+1.$$

Thus by the inductive hypothesis, k has a prime factor and so $n+1$ must have that same prime factor.

An example of double induction

Template:

$$P(m_0,n_0) \wedge ((n_0 \leq n) \Rightarrow (P(m_0,n) \Rightarrow P(m_0,n+1))) \wedge ((m_0 \leq m \wedge n_0 \leq n) \Rightarrow (P(m,n) \Rightarrow P(m+1,n)))$$

or

$$P(m_0,n_0) \wedge ((m_0 \leq m) \Rightarrow (P(m,n_0) \Rightarrow P(m+1,n_0))) \wedge ((m_0 \leq m \wedge n_0 \leq n) \Rightarrow (P(m,n) \Rightarrow P(m,n+1)))$$

Notice the first version does the final induction in the first parameter: m and the second version does the final induction in the second parameter: n. Thus, the "basis induction step" (i.e. the one in the middle) is also different in the two versions.

By double induction, I will prove that for $m,n \geq 1$

$$\sum_{i=1}^{m}(\sum_{j=1}^{n} ij) = \frac{mn(m+1)(n+1)}{4}.$$

For $m,n \geq 1$, let $P(m,n) =$ "$\sum_{i=1}^{m}(\sum_{j=1}^{n} ij) = \frac{mn(m+n+2)}{2}$".

First basis step: $P(1,1)$ is true since $\sum_{i=1}^{1}(\sum_{j=1}^{1} ij) = \sum_{i=1}^{1}(\sum_{j=1}^{1} 1) = 1 = \frac{1 \cdot 1(1+1)(1+1)}{4}$.

Inductive basis step for $n=1$: For $m \geq 1$, $P(m,1) \Rightarrow P(m+1,1)$, since if $\sum_{i=1}^{m}(\sum_{j=1}^{1} ij) = \frac{m(m+1)\cdot 2}{4} = \frac{m(m+1)}{2}$, then

$$\begin{aligned}
\sum_{i=1}^{m+1}(\sum_{j=1}^{1} ij) &= \sum_{i=1}^{m}(\sum_{j=1}^{1} ij) + \sum_{j=1}^{1}(m+1)\cdot 1 \\
&= \frac{m(m+1)}{2} + (m+1) \\
&= \frac{(m+1)((m+1)+1)\cdot 2}{4}.
\end{aligned}$$

Inductive step: For $m,n \geq 1$, $P(m,n) \Rightarrow P(m,n+1)$, since if $\sum_{i=1}^{m}(\sum_{j=1}^{n} ij) = \frac{mn(m+1)(n+1)}{4}$,

then

$$
\begin{aligned}
\sum_{i=1}^{m}(\sum_{j=1}^{n+1} ij) &= \sum_{i=1}^{m}(\sum_{j=1}^{n} ij + (i(n+1))) \\
&= \frac{mn(m+1)(n+1)}{4} + (n+1)\sum_{i=1}^{m} i \\
&= \frac{mn(m+1)(n+1)}{4} + (n+1)\frac{m(m+1)}{2} \\
&= \frac{mn(m+1)(n+1)}{4} + \frac{2m(m+1)(n+1)}{4} \\
&= \frac{m(n+1)(m+1)((n+1)+1)}{4}
\end{aligned}
$$