The equivalence of weak and strong induction

Imagine taking your sheet of Causey rules and striking out Modus Tolens. Would you loose anything? Well, if you knew $j \Rightarrow y$ and $\sim y$, then from the Contrapositive form $j \Rightarrow y$ could be written as $\sim y \Rightarrow \sim j$, and with Modus Ponens on this and $\sim y$, you could conclude $\sim j$. The result is that we don't need Modus Ponens – it's a handy short cut but anything that is proved **with** it can also be proved **without** it. You will find many Causey rules are like that: handy but not essential.

So what is the story on weak and strong induction? Does the strong induction axiom allow us to prove something that weak induction does not allow? The answer is no. Strong induction is simply a convenience: it gives no additional strength to our ability to prove.

Suppose we see the Weak Induction Template as:

$$P(n_0) \wedge \forall n((n \geq n_0) \Rightarrow (P(n) \Rightarrow P(n+1)))$$
$$\Rightarrow$$
$$\forall n((n \geq n_0) \Rightarrow P(n))$$

(You may be spooked by two unusual uses of implication in there and choose to write it as

$$P(n_0) \wedge \forall n \geq n_0 (P(n) \Rightarrow P(n+1))$$
$$\Rightarrow$$
$$\forall n \geq n_0 P(n)$$

Convince yourself that they actually are identical. The second form is easier to think about but notice the predicate $n \geq n_0$ is embedded in the set definition in the second case. Thus, we are using the set of natural numbers greater than or equal to n_0. That's fine but because of the template for which I want to state strong induction I am going to use the first form. The result is the set over which I take **every** universal quantification is the set of natural numbers.)

Now consider the Strong Induction Template as:

$$P(n_0) \wedge \forall n(\forall i((n \geq i \geq n_0) \Rightarrow P(i)) \Rightarrow P(n+1))$$
$$\Rightarrow$$
$$\forall n(n \geq n_0 \Rightarrow P(n))$$

(The easier to read version is written as

$$P(n_0) \wedge \forall n \geq n_0 (\forall i \ni n_o \leq i \leq n\ P(i) \Rightarrow P(n+1))$$
$$\Rightarrow$$
$$\forall n \geq n_0 P(n)$$

It's the $\forall i \ni n_o \leq i \leq n$ that I am trying to avoid. Yes, we understand what it means but I want this presentation to be bullet proof. If you can stomach "…" then you could write the strong induction template as:

$$P(n_0) \wedge \forall n \geq n_0 ((P(n_0) \wedge ... \wedge P(n)) \Rightarrow P(n+1))$$
$$\Rightarrow$$
$$\forall n \geq n_0 P(n).$$

That's way too loose for what I want to do.)

The key to showing the equivalence is to introduce a sort of "super-premise" Q. To be precise for $n \geq n_0$

$$Q(n) = \forall i((n \geq i \geq n_0) \Rightarrow P(i))$$

(The children's version of this is $Q(n) = (P(n_0) \wedge ... \wedge P(n))$. $Q(n)$ is no more than the conjunction of $P(n_0)$ through $P(n)$. Isn't it unfortunate that we have to write it in that other form? The reason is that too much can be lost in "…" . Without question many published proofs use the dots. Certainly it can be much easier to follow. The difficulty is the rigor: can one really present a proof without them? If not, it's worrisome.)

So I want to show you that proving "$\forall n \geq n_0 P(n)$" with strong induction can actually be accomplished by using weak induction on the Q's. If I can do that, then assuming you had a weak induction tool in your axiomatic tool chest, you could build yourself a strong induction tool. From then on use whichever one seems to fit. (If you like, imagine that your tool chest had Modus Ponens and Contrapositive. You could build Modus Tolens from them and use it when it seemed convenient. It didn't really give you any extra proving power – just convenience.)

In this manner suppose we were able to establish $Q(n_0) \wedge \forall n((n \geq n_0) \Rightarrow (Q(n) \Rightarrow Q(n+1)))$. What would we have done? Well, first we would have to prove the base case: $Q(n_0)$. But notice that $Q(n_0) = P(n_0)$, so the base case of the strong induction on the P's is that same as the base case for the weak induction on the Q's. (Yes, it's also the base case for the weak induction on the P's but that's not what we're dealing with here. Once again, we seek to show that proving "$\forall n \geq n_0 P(n)$" with strong induction on the P's can actually be accomplished by using weak induction on the Q's.)

Now move to the inductive step. Suppose I prove $\forall n((n \geq n_0) \Rightarrow (Q(n) \Rightarrow Q(n+1)))$. For a moment think of that in the "…" form:

$$\forall n((P(n_0) \wedge ... \wedge P(n)) \Rightarrow (P(n_0) \wedge ... \wedge P(n) \wedge P(n+1))).$$

Part of that seems silly: Clearly on the right hand side the portion $P(n_0) \wedge ... \wedge P(n)$ comes for free since you've assumed it on the left hand side. The only thing we care about establishing is the last guy: $P(n+1)$. Let's rewrite it then in the equivalent fashion excluding the junk terms $P(n_0) \wedge ... \wedge P(n)$ on the right:

$$\forall n((P(n_0) \wedge ... \wedge P(n)) \Rightarrow P(n+1)).$$

Oops. Isn't that just strong induction on the P's? Of course it is. Let's tidy it up without using "…".

Substituting "$\forall i((n \geq i \geq n_0) \Rightarrow P(i))$" for $Q(n)$ in the **weak** inductive step on the Q's

$$\forall n((n \geq n_0) \Rightarrow (Q(n) \Rightarrow Q(n+1)))$$

we get for the P's

$$\forall n((n \geq n_0) \Rightarrow (\forall i((n \geq i \geq n_0) \Rightarrow P(i)) \Rightarrow \forall i((n+1 \geq i \geq n_0) \Rightarrow P(i))).$$

If $\forall i((n+1 \geq i \geq n_0) \Rightarrow P(i))$ is true (i.e., all of the $P(i)$'s for i from n_0 to $n+1$ are true) then certainly $P(n+1)$ alone is true. Also notice that having an i satisfying $n \geq i \geq n_0$ guarantees that $n \geq n_0$ so we could drop the $n \geq n_0$ assumption. We could then say we had proved

$$\forall n(\forall i((n \geq i \geq n_0) \Rightarrow P(i)) \Rightarrow P(n+1)).$$

Wonderful, because that's precisely the **strong** inductive step using the P's.