Checking for Transitivity and Computing Transitive Closures

Let $A = \{a_1, a_2, ..., a_n\}$ be a set of cardinality n and R be a relation on A. We seek here to find simple ways to determine if R is transitive and to compute the transitive closure. Notice that if we were to compute the transitive closure and if at any point we had to augment R then we would know that the original relation was not transitive. Put in other words, the algorithm for transitive closure could serve as an algorithm for determining transitivity. Nevertheless, it may be a little bit easier to discuss them separately – at least initially.

The definition of transitivity says R is transitive if and only if

$$\forall x \in A(\forall y \in A(\forall z \in A(((x,y) \in R \wedge (y,z) \in R) \Rightarrow (x,z) \in R).$$

It will seem to be a trivial change but let's rearrange the order of the quantifiers to place an emphasis on y

$$\forall y \in A(\forall x \in A(\forall z \in A(((x,y) \in R \wedge (y,z) \in R) \Rightarrow (x,z) \in R).$$

Now consider this for a fixed element $y \in A$: suppose I know that every element x related to y (i.e., $(x,y) \in R$) and every element z to which y is related (i.e., $(y,z) \in R$) are related to each other (i.e., $(x,z) \in R$), then I know that at least transitivity hold for pairs of relations "with y in the middle". I would have actually transitivity if I were able to assert this for all $y \in A$. This is the basis for the algorithm. All triples (y,x,z) are checked.

The following algorithm (sometimes called Warshall's Algorithm or the Roy-Warshall Algorithm consistent with a tradition in computer science that trivial results get names attached) can be used to find the transitive closure $\bar{R}$ of a relation R. We will actually keep track of the relations through the incidence matrices $\bar{M}$ and M, respectively.

```
Set M̄ = M .
For k = 1,...,n
    For i = 1,...,n
            If M̄_{i,k} = 1 then
                    For j = 1,...,n
                            If M̄_{k,j} = 1 then set M̄_{i,j} = 1
```

(If we must make a big deal about what is and isn't Warshall's Algorithm then it should be added that this is my improvement to the algorithm.)

Lastly, a slight variant can be used for transitivity checking. If the algorithm never finds R is not transitive, then R is transitive.

```
For k = 1,...,n
    For i = 1,...,n
            If M_{i,k} = 1 then
                    For k = 1,...,n
                            If M_{k,j} = 1 and M_{i,j} = 0 then R is not transitive STOP.
```