1. Using induction, prove that for $r \neq 0,1$ and $n \geq 0$, $\sum_{k=0}^{n} r^k = \dfrac{r^{n+1}-1}{r-1}$.

For $n \geq 0$, let $P(n)$ = "$\sum_{k=0}^{n} r^k = \dfrac{r^{n+1}-1}{r-1}$".

Basis step: $P(0)$ is true since $\sum_{k=0}^{0} r^k = 1 = \dfrac{r^{0+1}-1}{r-1}$..

Inductive step: For $n \geq 1$, $P(n) \Rightarrow P(n+1)$, since if $\sum_{k=0}^{n} r^k = \dfrac{r^{n+1}-1}{r-1}$, then

$$\begin{aligned}\sum_{k=0}^{n+1} r^k &= \sum_{i=0}^{n} r^k + r^{n+1} \\ &= \frac{r^{n+1}-1}{r-1} + \frac{r^{n+2}-r^{n+1}}{r-1} \\ &= \frac{r^{(n+1)+1}-1}{r-1}.\end{aligned}$$

2. Consider the Fibonacci sequence: $f_0 = 1$, $f_1 = 1$, $f_k = f_{k-1} + f_{k-2}$, for $k \geq 2$. Using induction, prove that for $n \geq 0$, $\sum_{k=0}^{n} f_k^2 = f_n f_{n+1}$.

For $n \geq 0$, let $P(n)$ = "$\sum_{k=0}^{n} f_k^2 = f_n f_{n+1}$".

Basis step: $P(0)$ is true since $\sum_{k=0}^{n} f_k^2 = f_0^2 = 1 = 1 \cdot 1 = f_0 f_1$..

Inductive step: For $n \geq 0$, $P(n) \Rightarrow P(n+1)$, since if $\sum_{k=0}^{n} f_k^2 = f_n f_{n+1}$, then

$$\begin{aligned}\sum_{k=0}^{n+1} f_k^2 &= \sum_{k=0}^{n} f_k^2 + f_{n+1}^2 \\ &= f_n f_{n+1} + f_{n+1}^2 \\ &= f_{n+1}(f_n + f_{n+1}) \\ &= f_{n+1} f_{(n+1)+1}.\end{aligned}$$

3. Using induction prove for $n \geq 2$, that $\prod_{k=2}^{n} (1 - \dfrac{1}{k^2}) = \dfrac{n+1}{2n}$.