Cardinality Theory

Definition 1: A set *A* is *finite* with cardinality n if it is empty or if there exists a one-to-one function mapping $\{1,2,\ldots,n\}$ onto *A*. A set is *infinite* if it is not finite.

Definition 2: A set *A* is *infinite* if there exists a one-to-one function mapping *A* onto a proper subset of *A*. A set is *finite* if it is not infinite.

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Theorem 1: *The set* $\mathbf{N}$ *of natural numbers is infinite.*

Theorem 2: *The real interval* $[0,1]$ *is infinite.*

Theorem 3: *A superset of an infinite set is infinite.*

Corollary: *A subset of a finite set is finite.*

Theorem 4: *Let A be infinite and* $f: A \xrightarrow{1-1} B$, *then B is infinite.*

Definition 3: A set *A* is *countably infinite* if there exists a one-to-one function mapping N onto *A*. A set is *countable* if it is finite or countably infinite. A set is *uncountably infinite* if it is not countable.

Theorem 5: *The real interval* $[0,1]$ *is uncountably infinite.*

Theorem 6: *The set of infinitely long bit strings is uncountably infinite.*

Theorem 7: *If there exists a function* $f: \mathrm{N} \xrightarrow[onto]{} A$ *then* A *is countable.*

Theorem 8: *A subset of a countable set is countable.*

Corollary 8.1: *A superset of an uncountably infinite set is uncountably infinite.*

Theorem 9: *The union of a finite collection of finite sets is finite.*

Theorem 10: *The union of a countably infinite collection of finite sets is countable.*

Theorem 11: *The union of a countably infinite collection of countably infinite sets is countably infinite.*

Corollary 11.1: *The union of a finite collection of countably infinite sets is countably infinite.*

Corollary 11.2: *The union of a countable collection of countably infinite sets is countably infinite.*

Corollary 11.3: *If the set* A *is countably infinite and the set* B *is countable then the Cartesian product* $A \times \mathrm{B}$ *is countably infinite.*

Theorem 12: *Let* A *be uncountably infinite and* $f: A \xrightarrow{1-1} B$, *then* B *is uncountably infinite.*