Asymptotic Dominance Theory

• **Definition 1**: Given the functions \( f : \mathbb{N} \to \mathbb{R} \) and \( g : \mathbb{N} \to \mathbb{R} \), \( f \) is **asymptotically dominated** by \( g \) if there exist non-negative constants \( M \) and \( N \) such that for all \( n \geq N \), \(|f(n)| \leq M|g(n)|\). This is denoted by \( f = O(g) \).

• **Definition 2**: Given the functions \( f : \mathbb{N} \to \mathbb{R} \) and \( g : \mathbb{N} \to \mathbb{R} \), \( f = o(g) \) if for every positive \( \varepsilon \), there exists a non-negative constant \( N \) such that for all \( n \geq N \), \(|f(n)| \leq \varepsilon |g(n)|\).

**Theorem 1**: If \( f = O(g) \), then for any constant \( s \), \( sf = O(g) \).

**Proof**: By definition, there exist non-negative constants \( M \) and \( N \) such that for all \( n \geq N \), \(|f(n)| \leq M|g(n)|\). Thus for all \( n \geq N \), \(|sf(n)| \leq |s|M|g(n)|\). Therefore, \( sf = O(g) \).

**Theorem 2**: If \( f_1 = O(g_1) \) and \( f_2 = O(g_2) \), then \( f_1 + f_2 = O(|g_1| + |g_2|) \).

**Proof**: By definition, there exist non-negative constants \( M_1 \) and \( N_1 \) such that for all \( n \geq N_1 \), \(|f_1(n)| \leq M_1|g_1(n)|\) and there exist non-negative constants \( M_2 \) and \( N_2 \) such that for all \( n \geq N_2 \), \(|f_2(n)| \leq M_2|g_2(n)|\). For \( n \geq \max\{N_1, N_2\} \) both inequalities hold so \(|f_1(n) + f_2(n)| \leq |f_1(n)| + |f_2(n)| \leq M_1|g_1(n)| + M_2|g_2(n)| \leq \max\{M_1, M_2\}|g_1(n)| + |g_2(n)| \). Therefore, \( f_1 + f_2 = O(|g_1| + |g_2|) \).

**Corollary 2.1**: If for \( i = 1, 2, \ldots, k \), \( f_i = O(g_i) \), then \( \sum_{i=1}^{k} f_i = O(\sum_{i=1}^{k} |g_i|) \).

**Theorem 3**: If \( f_1 = O(g_1) \) and \( f_2 = O(g_2) \), then \( f_1 + f_2 = O(\max\{|g_1|, |g_2|\}) \).

**Proof**: By definition, there exist non-negative constants \( M_1 \) and \( N_1 \) such that for all \( n \geq N_1 \), \(|f_1(n)| \leq M_1|g_1(n)|\) and there exist non-negative constants \( M_2 \) and \( N_2 \) such that for all \( n \geq N_2 \), \(|f_2(n)| \leq M_2|g_2(n)|\). For \( n \geq \max\{N_1, N_2\} \) both inequalities hold so \(|f_1(n) + f_2(n)| \leq |f_1(n)| + |f_2(n)| \leq M_1|g_1(n)| + M_2|g_2(n)| \leq (M_1 + M_2) \max\{|g_1(n)|, |g_2(n)|\} \). Therefore, \( f_1 + f_2 = O(\max\{|g_1|, |g_2|\}) \).

**Corollary 3.1**: If for \( i = 1, 2, \ldots, k \), \( f_i = O(g_i) \), then \( \sum_{i=1}^{k} f_i = O(\max_{i=1,\ldots,k}|g_i|) \).

**Corollary 3.2**: If for \( i = 1, 2, \ldots, k \), \( f_i = O(g) \), then \( \sum_{i=1}^{k} f_i = O(g) \).

**Theorem 4**: If \( f_1 = O(g_1) \) and \( f_2 = O(g_2) \), then \( f_1 \cdot f_2 = O(g_1 \cdot g_2) \).
Proof: By definition, there exist non-negative constants $M_1$ and $N_1$ such that for all $n \geq N_1$, $|f_1(n)| \leq M_1 |g_1(n)|$ and there exist non-negative constants $M_2$ and $N_2$ such that for all $n \geq N_2$, $|f_2(n)| \leq M_2 |g_2(n)|$. For $n \geq \max\{N_1, N_2\}$ both inequalities hold so $|f_1(n) \cdot f_2(n)| = |f_1(n)| \cdot |f_2(n)| \leq M_1 |g_1(n)| \cdot M_2 |g_2(n)| \leq (M_1 \cdot M_2) (|g_1(n)| \cdot |g_2(n)|)$. Therefore, $f_1 \cdot f_2 = O(g_1 \cdot g_2)$.

Corollary 4.1: If for $i = 1, 2, \ldots, k$, $f_i = O(g_i)$, then $\prod_{i=1}^{k} f_i = O(\prod_{i=1}^{k} g_i)$.

Theorem 5: If $f_1 = O(g_1)$, $g_2 = O(f_2)$, and $g_2$ has no zeros. then $f_1 / f_2 = O(g_1 / g_2)$.

Proof: By definition, there exist non-negative constants $M_1$ and $N_1$ such that for all $n \geq N_1$, $|f_1(n)| \leq M_1 |g_1(n)|$ and there exist non-negative constants $M_2$ and $N_2$ such that for all $n \geq N_2$, $|g_2(n)| \leq M_2 |f_2(n)|$. Notice that since $g_2$ has no zeros, then neither does $f_2$. Inverting this inequality, we obtain that for all $n \geq N_2$, $|1 / f_2(n)| \leq M_2 |1 / g_2(n)|$. For $n \geq \max\{N_1, N_2\}$ both inequalities hold so $|f_1(n) / f_2(n)| = |f_1(n)| \cdot |1 / f_2(n)| \leq M_1 |g_1(n)| \cdot M_2 |1 / g_2(n)| \leq (M_1 \cdot M_2) (|g_1(n)| / |g_2(n)|)$. Therefore, $f_1 / f_2 = O(g_1 / g_2)$.

Theorem 6: If $a \leq b$, then $n^a = O(n^b)$.

Proof: For $n \geq 0$, $n^{-(b-a)} \leq n^0 = 1$, and $|n^a| = |n^{-(b-a)} n^b| = |n^{-(b-a)}| n^b | \leq 1 \cdot n^b$. Therefore, $n^a = O(n^b)$.

Theorem 7: If $a < b$, then $n^a = o(n^b)$.

Proof: Given any $\varepsilon > 0$, let $N = (1 / \varepsilon)^{(b-a)}$. Notice then for $n \geq N = (1 / \varepsilon)^{(b-a)}$, $n^{b-a} \geq 1 / \varepsilon$, and $n^{-(b-a)} \leq \varepsilon$. So $|n^a| = |n^{-(b-a)} n^b| = |n^{-(b-a)}| n^b | \leq \varepsilon | n^b |$. Therefore, $n^a = o(n^b)$. 

Example 1: If $f_1 = O(g_1)$ and $f_2 = O(g_2)$, then $f_1 / f_2$ may not be $O(g_1 / g_2)$.

Proof: Let $f_1(n) = f_2(n) = 1$, for all $n \geq 0$. Then $f_1 = O(1)$ and $f_2 = O(n)$ but $f_1 / f_2 = 1 \neq O(1/n)$. To see this, consider any $N \geq 0$ and $M \geq 0$. Choose any $n > \max\{N, M\}$. Notice that then $|1| = 1 > M / |n|$, so $1 \neq O(1/n)$.

Example 2: If $a < b$, then $n^b \neq O(n^a)$.

Proof: Suppose $n^b = O(n^a)$, then there exist $N \geq 0$ and $M \geq 0$ so that for all $n \geq N$, $|n^b| \leq M |n^a|$. Choose any $n > \max\{N, M^{(b-a)}\}$. Notice that then $n^{b-a} > M$, so $|n^b| = n^b > M n^a = M |n^a|$, and $n^b \neq O(n^a)$.

Example 3: If $f$ is any polynomial of degree $k$ then $f = O(n^k)$.

Proof-1: Without loss of generality, assume $f(n) = \sum_{j=0}^{k} a_i n^i$. For all $n \geq 0$ and $0 \leq i \leq k$, $|n^i| \leq |n^k|$ and $|a_i n^i| \leq |a_i| \cdot |n^k|$. So for $N = 0$ and $M = \sum_{i=0}^{k} |a_i|$, we have $n \geq N$ implies $|f(n)| = \left| \sum_{i=0}^{k} a_i n^i \right| \leq \sum_{i=0}^{k} |a_i n^i| \leq \sum_{i=0}^{k} |a_i| \cdot |n^k| \leq \left( \sum_{i=0}^{k} |a_i| \right) \cdot |n^k| = M |n^k|$

Proof-2: Without loss of generality, assume $f(n) = \sum_{j=0}^{k} a_i n^i$. By Theorem 6, $n^i = O(n^k)$, for $0 \leq i \leq k$. By Theorem 1, $a_i n^i = O(n^k)$ for $0 \leq i \leq k$. Finally, from Corollary 3.2, $f(n) = \sum_{i=0}^{k} a_i n^i = O(n^k)$. 