Examination 1

CS 336

1. [5] Consider integers in the set {1, 2, 3, …, 1000}. How many are divisible by either 4 or 10?

Let D_4 = {*set of integers between 1 and 1000 divisible by 4*} and let D_{10} = {*set of integers between 1 and 1000 divisible by 10*}. We have:

$$\begin{aligned}\#(D_4 \cup D_{10}) &= \#(D_4) + \#(D_{10}) - \#(D_4 \cap D_{10}) \\ &= 250 + 100 - 25 \\ &= 325.\end{aligned}$$

2. a. [10] Present a combinatorial argument that for all $n \geq 1$:

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$$

Let A = {*a, b, c*} and consider all strings of length n using elements of A. Since there are three options for each component of the string, there are 3^n such strings. Alternatively, consider first consider the positions of any *c*'s in the string. Let *k* represent the number of non-*c*'s (i.e., *a*'s and *b*'s) in the string. Clearly *k* could range from 0 through *n*. For a fixed value of *k*, there are $\binom{n}{k}$ ways to choose the positions for the non-*c*'s . Then for each of the *k* positions, there are two options (i.e., *a* or *b*) for the character in the position. The remaining *n*-*k* positions must be occupied by *c*'s. Thus there are $\binom{n}{k}2^k$ ways to assign elements to the positions with *k* non-*c*'s. The total is $\sum_{k=0}^{n}\binom{n}{k}2^k$ and this must equal 3^n

b. [10] Present a combinatorial argument that for all nonnegative integers *p*, *s*, and *n* satisfying $p + s \leq n$

$$\binom{n}{p}\binom{n-p}{s} = \binom{n}{p+s}\binom{p+s}{p}$$

(Hint: Consider choosing two subsets.)

Let a set A have *n* elements and consider how many ways there are to select disjoint subsets *B* and *C* of A so that *B* has p elements and *C* has *s* elements. First we could select the *p* elements for *B* in $\binom{n}{p}$ ways and then select the *s* elements for *C* from the remaining *n-p* elements of $A \sim B$ in $\binom{n-p}{s}$ ways. Together this yields $\binom{n}{p}\binom{n-p}{s}$ such selections. Alternatively, we could first select the *p+s* elements for $B \cup C$ in $\binom{n}{p+s}$ ways and then select the p elements for *B* from $B \cup C$ in $\binom{p+s}{p}$ ways. There are thus $\binom{n}{p+s}\binom{p+s}{p}$ such selections and this must equal $\binom{n}{p}\binom{n-p}{s}$

3. [10] For $n \geq 1$, Let A = {1, 2, …, 2*n*}. How many subsets of A contain exactly k_1 even numbers and k_2 odd numbers?

There are *n* even numbers and *n* odd numbers in the set A. Thus there are $\binom{n}{k_1}\binom{n}{k_2}$ such subsets.

4. [10] For $n \geq 1$, how many ordered triples (n_1, n_2, n_3) of non-negative numbers satisfy $n_1 + n_2 + n_3 = n$? (Hint: think about putting balls into bins.)

There are $\binom{n+3-1}{3-1}$ arrays of *n* unordered components composed of elements selected from {1, 2, 3} allowing repetition.. For a given array, let n_1 = the number of 1's, n_2 = the number of 2's, and n_3 = the number of 3's. Since the array is of length *n*, $n_1 + n_2 + n_3 = n$ and every decomposition of n into $n_1 + n_2 + n_3$ must be associated with a unique such array. Thus there are $\binom{n+2}{2}$ such decompositions.

5. [10] For $n \geq 1$, assume all strings of *n* characters from $\{a, b, c, d\}$ are equally likely. What is the expected number of *a*'s in such a string?

There are 4^n equally likely strings. Let k represent the number of *a*'s in the string. The value of k can range from 0 through *n*. For a fixed k, there are $\binom{n}{k}$ ways to position the *a*'s and then 3^{n-k} ways to assign the other three characters into the remaining *n-k* positions. The total is $\binom{n}{k}3^{n-k}$ and thus the probability is $\binom{n}{k}3^{n-k}/4^n$. The expected number of *a*'s is then $\sum_{k=0}^{n}\binom{n}{k}3^{n-k}/4^n$.

6. [10] Given a finite event space E (in which all events are equally likely) and subsets A and B of E, show that

$$\Pr(A \cup B) \geq \Pr(A) + \Pr(B) - 1.$$

Since $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$ and $\Pr(A \cap B) \leq 1$, we have $-\Pr(A \cap B) \geq -1$ and $\Pr(A \cup B) \geq \Pr(A) + \Pr(B) - 1$

7. [10] Consider a 52 card deck of cards from which the ace of spades is removed resulting in a 51 card deck. Further, consider two distinct cards drawn from the 51 card deck and assume all such unordered draws are equally likely. Lastly consider the probability of the event that both cards are hearts. Is it more likely that both cards are hearts if it is given that both cards are face cards (i.e., Kings, Queens, or Jacks)?

There are $\binom{51}{2}$ equally likely hands. Of these $\binom{13}{2}$ are all hearts, thus the probability of an all heart hand is $\binom{13}{2}/\binom{51}{2} = 78/1275 = .061...$ The probability of both card being face cards is $\binom{12}{2}/\binom{51}{2}$ since there are 12 face cards. The probability of having all hearts and all face cards is $\binom{3}{2}/\binom{51}{2}$ so the probability of having two hearts given two face cards is $\dfrac{\binom{3}{2}/\binom{51}{2}}{\binom{12}{2}/\binom{51}{2}} = \binom{3}{2}/\binom{12}{2} = 3/66 = .0454...$. So it is less likely that that both cards are hearts if it is given that both cards are face cards

8. [10] Let A be a set of cardinality p. Consider ordered strings of length m using the elements of A. How many such strings have the m^{th} component a repetition of one of the preceding m-1? (Hint: Think about the complement and think about selecting the m^{th} component first.)

There are p^m strings of length m. There are p choices for the m^{th} component and p-1 choices for each of the preceding components if each of the preceding m-1 components differ from the m^{th}. Thus there are $p \cdot (p-1)^{m-1}$ such strings in which the m^{th} component differs from all predecessors and finally $p^m - p \cdot (p-1)^{m-1}$ strings in which the m^{th} component is a repetition of one of the preceding m-1.