Examination 1 Solutions

CS 336

1. The important issue is the logic you used to arrive at your answer.
2. Use extra paper to determine your solutions then neatly transcribe them onto these sheets.
3. Do not submit the scratch sheets. However, all of the logic necessary to obtain the solution should be on these sheets.
4. Comment on all logical flaws and omissions and enclose the $\boxed{\textbf{comments in boxes.}}$

1. [5] For $n \geq 3$, how many subsets of size 3 from $\{a_1, a_2, \ldots, a_n\}$ are there that either contain a_1 or a_2 (or both)?

There are $\binom{n}{3}$ subsets of $\{a_1, a_2, \ldots, a_n\}$ of size 3. If $n = 3$ or $n = 4$ all subsets contain a_1 or a_2 (or both) so the answer is $\binom{n}{3}$. However for $n \geq 5$, there are $\binom{n-2}{3}$ subsets that avoid both a_1 and a_2, thus $\binom{n}{3} - \binom{n-2}{3}$ contain a_1 or a_2 (or both).

2. [10] Given a set A of m characters, for $n \geq 2$, consider strings of length n using any of the characters of A. How many such strings begin and end with the same character?

There are m^{n-1} strings of length $n-1$ using the characters of A. Since the nth character must agree with the first, this is also the number of strings of length n that begin and end with the same character.

3. [10] Present a combinatorial argument that for all positive integers $m, n,$ and r, satisfying $r \leq \min\{m, n\}$:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}.$$

(**Hint**: Consider selecting from two sets.)

Let A and B be disjoint sets of cardinalities m and n, respectively. Let $C = A \cup B$ and consider the number of subsets of C of cardinality r. Since $|C| = |A| + |B) = m + n$, there are $\binom{m+n}{r}$ such subsets. Alternatively let k be the number of elements in a subset that came from A. The value of k can range from 0 to r. For a fixed value of k, there are $\binom{m}{k}$ ways to select the k elements from A and $\binom{n}{r-k}$ ways to select the remaining $r-k$ elements from B, thus $\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k}$ total ways. This must equal $\binom{m+n}{r}$.

b. [10] Present a combinatorial argument that for all positive integers n:

$$3^n = \sum_{i=0}^{n}\left(\sum_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}\right)$$

(**Note:** Be very specific about the roles of i and j.)

Consider the strings of length n using the characters from $\{1,2,3\}$. Since each position has three options there are 3^n such strings. Alternatively, let i be the number of 1's in the string and j be the number of 2's. The value of i can range from 0 to n and, once that is fixed, the value of j can range from 0 to $n-i$. For a string with i 1's and j 2's, there are $\binom{n}{i}$ ways to locate the i 1's in the n positions and then $\binom{n-i}{j}$ ways to locate the j 2's in the remaining $n-i$ positions. The $n-i-j$ positions still left must contain 3's. Totally this we have $\sum_{i=0}^{n}\left(\sum_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}\right)$ such strings and this must equal 3^n.

4. [10] How many distinct permutations are there of the digits in 1121231234?

Since there are 4 1's, 3 2's, 2 3's, and 1 4, there are 10 total characters and $\binom{10}{4\quad 3\quad 2}$ permutations.

5. [10] Given $n \geq r \geq 1$, in how many ways can n identical balls be placed into r distinct bins such that each bin contains at least one ball? (Hint: Consider strings with balls and special dividers.)

We may consider that we begin with one ball already positioned in each of the three bins and thus $n-3$ balls remain. There are $\binom{n-3+2}{2}$

6. [10] For $n \ge 1$, consider strings of length $2n$ 0's and 1's. Assuming all such strings are equally likely, what is the probability that such a string has an equal number of 0's and 1's?

There are 2^{2n} equally likely bit strings of length $2n$. For there to be an equal number of 0's and 1's, there must be n of each. There are $\binom{2n}{n}$ such strings, so the probability of having a bit string with equal number of 0's and 1's is $\binom{2n}{n}/2^{2n}$.

7. a. [10] For $n \ge 5$, consider strings of length n using elements of $\{a,b,c,d\}$. Assume all such strings are equally likely. What is the probability that a string has exactly two *a*'s?

There are 4^n equally strings of length n using elements of $\{a,b,c,d\}$. For a string to have exactly two *a*'s, there are $\binom{n}{2}$ ways to select the positions for the *a*'s and then 3^{n-2} ways to complete the string using the other three characters. Thus there are $\binom{n}{2}3^{n-2}$ such strings and the probability of having a string with exactly two *a*'s is $\binom{n}{2}3^{n-2}/4^n$.

b. [5] What is the probability that such a string has exactly three *b*'s given that it has exactly two *a*'s?

For a string to have exactly three *b*'s and exactly two *a*'s, there are $\binom{n}{3}$ ways to select the positions for the *b*'s, $\binom{n-3}{2}$ ways to select the positions for the *a*'s and then 2^{n-5} ways to complete the string using the other two characters. Thus there are $\binom{n}{3}\binom{n-3}{2}2^{n-5}$ such strings The probability of having exactly three *b*'s given that it has exactly two *a*'s is $\binom{n}{3}\binom{n-3}{2}2^{n-5}/\binom{n}{2}3^{n-2}$.