1. [20] Using only Definition 2', show that the set of negative integers is infinite.

Let $A$ be the set of negative integers and define $f : A \to A$ by $f(a) = a - 1$ for $a \in A$. If $a_1, a_2 \in A$ and $a_1 \neq a_2$ then $f(a_1) = a_1 - 1 \neq a_2 - 2 = f(a_2)$, so $f$ is one to one. The integer $-1$ is negative but for no negative integer $a$ is $a - 1 = -1$ so $f$ maps $A$ to a proper subset of itself and $A$ is infinite.

2. [20] Suppose the set $A$ is uncountably infinite, the set $B$ is countably infinite, and the set $C$ is finite. Let $D = A \cup B \cup C$. Is $D$ finite, countably infinite, or uncountably infinite? Prove your claim.

The set $D = A \cup B \cup C$ is a superset of $A$. By Corollary 9.1 it is uncountably infinite.

3. [20] Suppose the set $A$ is non-empty and the set $B$ is uncountably infinite. Prove that the cartesian product $A \times B$ is uncountably infinite.

Choose $\bar{a} \in A$ and define $f : B \to A \times B$ by $f(b) = (\bar{a}, b)$ for all $b \in B$. We see $f$ is one-to-one since for $b_1 \neq b_2$, $f(b_1) = (\bar{a}, b_1) \neq (\bar{a}, b_2) = f(b_2)$. By Theorem 10, $A \times B$ is uncountably infinite.

4. [20] Using only Definition 1, prove that $3n^4 = O(n^{4.5})$.

Let $M = 3$ and $N = 1$. For $n \geq N = 1$, we have $\sqrt{n} \geq 1$, so $3n^4 \leq 3 \cdot n^4 \cdot \sqrt{n} = 3 \cdot n^{4.5}$. Thus $3n^4 = O(n^{4.5})$.

5. [20] Using only Definition 2, prove that $5^n \neq o(2 \cdot 4^n)$.

Let $\varepsilon = 1/4$ and suppose there exists $N$ so that for all $n \geq N$, $|5^n| \leq \varepsilon \cdot |2 \cdot 4^n|$. But for $n = \max\{1, \lceil N \rceil\}$, we have $n \geq N$ and $n \geq 1$, so $\left(\frac{5}{4}\right)^n > 1$ and $5^n > 4^n$, thus $|5^n| = 5^n > 4^n = 1/2 \cdot 2 \cdot 4^n = \varepsilon \cdot 2 \cdot 4^n$ and $5^n \neq o(2 \cdot 4^n)$.

6. [20] Suppose $f = O(g)$ and $g = O(h)$, prove or disprove (with a simple counterexample) that $f = O(h)$.

Suppose $f = O(g)$ and $g = O(h)$, then by definition, there exist $N_f \geq 0$, $M_f \geq 0$, $N_g \geq 0$, $M_g \geq 0$, so that for $n \geq N_f$, $|f(n)| \leq M_f |g(n)|$ and for $n \geq N_g$, $|g(n)| \leq M_g |h(n)|$. Thus for $n \geq \max\{N_f, N_g\}$, $|f(n)| \leq M_f M_g |h(n)|$. We may conclude that $f = O(h)$.