Examination 2 **Solutions**

1. **[20]** Using only Definition 2', show that the set of negative integers is infinite**.**

Let A be the set of negative integers and define $f: A \to A$ by $f(a) = a - 1$ for $a \in A$. If $a_1, a_2 \in A$ and $a_1 \neq a_2$ then $f(a_1) = a_1 - 1 \neq a_2 - 2 = f(a_2)$, so f is one to one. The integer –1 is negative but for no negative integer a is $a - 1 = -1$ so f maps A to a proper subset of itself and A is infinite.

2. **[20]** Suppose the set *A* is uncountably infinite, the set *B* is countably infinite, and the set *C* is finite. Let $D = A \cup B \cup C$. Is *D* finite, countably infinite, or uncountably infinite? Prove your claim.

The set $D = A \cup B \cup C$ is a superset of *A*. By Corollary 9.1 it is uncountably infinite.

3. **[20]** Suppose the set *A* is non-empty and the set *B* is uncountably infinite. Prove that the cartesian product $A \times B$ is uncountably infinite.

Choose $\overline{a} \in A$ and define $f: B \to A \times B$ by $f(b) = (\overline{a}, b)$ for all $b \in B$. We see *f* is one-to-one since for $b_1 \neq b_2$, $f(b_1) = (\overline{a}, b_1) \neq (\overline{a}, b_2) = f(b_2)$. By Theroem 10, $A \times B$ is uncountably infinite.

4. **[20]** Using only Definition 1, prove that $3n^4 = \mathrm{O}(n^{4.5})$.

Let $M = 3$ and $N = 1$. For $n \geq N = 1$, we have $\sqrt{n} \geq 1$, so $|3n^4| \leq 3\ n^4 \sqrt{n} = 3|n^{4.5}|$. Thus $3n^4 = \mathrm{O}(n^{4.5})$.

5. **[20]** Using only Definition 2, prove that $5^n \neq o(2 \cdot 4^n)$.

Let $\varepsilon = 1/4$ and suppose there exists N so that for all $n \geq N$, $|5^n| \leq \varepsilon |2 \cdot 4^n|$. But for $n = \max\{1, \lceil N \rceil\}$, we have $n \geq N$ and $n \geq 1$, so $(\frac{5}{4})^n > 1$ and $5^n > 4^n$, thus $|5^n| = 5^n > 4^n = 1/2 |2 \cdot 4^n| = \varepsilon |2 \cdot 4^n|$ and $5^n \neq o(2 \cdot 4^n)$.

6. **[20]** Suppose $f = \mathrm{O}(g)$ and $g = \mathrm{O}(h)$, prove or disprove (with a simple counter-example) that $f = \mathrm{O}(h)$.

Suppose $f = \mathrm{O}(g)$ and $g = \mathrm{O}(h)$, then by definition, there exist $N_f \geq 0,\ M_f \geq 0,\ N_g \geq 0,\ M_g \geq 0,$ so that for $n \geq N_f$, $|f(n)| \leq M_f |g(n)|$ and for $n \geq N_g$, $|g(n)| \leq M_g |h(n)|$. Thus for $n \geq \max\{N_f, N_g\}$, $|f(n)| \leq M_f M_g |h(n)|$. We may conclude that $f = \mathrm{O}(h)$.