1. The important issue is the logic you used to arrive at your answer.
2. Use extra paper to determine your solutions then neatly transcribe them onto these sheets.
3. Do not submit the scratch sheets. However, all of the logic necessary to obtain the solution should be on these sheets.
4. Comment on all logical flaws and omissions and enclose the comments in boxes

1. a. [5] How many strings of length \( n \geq 0 \) using characters \( a, b \) or \( c \) with possible repetition, have exactly \( n_a \) as and \( n_b \) bs (where \( n_a + n_b \leq n \) )?

b. [10] How many strings of length \( n \geq 0 \) using characters \( a, b \) or \( c \) with possible repetition, have either exactly \( n_a \) as or exactly \( n_b \) bs or both (where \( n_a + n_b \leq n \) )?

2. [10] For \( n \geq 1 \), how many four-tuples \( \langle i, j, k, l \rangle \) of non-negative values \( i, j, k, l \) satisfy \( i + j + k + l \leq n \)? (Hint: First consider the situation \( i + j + k + l = n \) and then think about \( m = n - (i + j + k + l) \).

3. a. [10] Using a combinatorial argument, prove that for \( n \geq 1 \) and \( m \geq 2 \):
\[
\sum_{k=0}^{n} \binom{n}{k} (m-1)^k = m^n
\]

b. [10] Using a combinatorial argument, prove that for \( n \geq k \geq 0 \):
\[
\binom{n}{k} k!(n-k)! = n!
\]

4. a. [10] For \( 3 \leq m \leq n \), what is the probability that a string of length \( m \) selected without repetition from \( \{1,2,\ldots,n\} \) contains the substring \( \langle 1,2,3 \rangle \)? (You may assume all strings of length \( m \) selected without repetition from \( \{1,2,\ldots,n\} \) are equally probable.)

5. [10] Using definition 2' (and no cardinality theorems) show that the set of reciprocals of positive integers (i.e., \( \{1/p \mid p \in \mathbb{Z} \land p > 0 \} \) ) is infinite.
6. a. [10] Let \( A = \{a,b,c,...,z,A,B,C,...,Z\} \) and and let \( B \) be the set of finite strings from \( A \), that is \( B = \{\alpha_1,\alpha_2,...,\alpha_n\} | n \in \mathbb{N} \land \alpha_i \in A \text{ for } i = 1,2,...,n \} \). Is the set \( B \) finite, countably infinite, or uncountably infinite? Prove your claim.

b. [5] Prove that the set of finite sets of real values from \([0,1]\)
\( C = \{\{x_1,x_2,...,x_n\} | n \in \mathbb{N} \land x_i \in [0,1] \text{ for } i = 1,2,...,n \} \) is uncountably infinite.

7. [10] Prove that if \( f_1 = O(g_1) \) and \( f_2 = o(g_2) \), then \( f_1f_2 = o(g_1g_2) \).

8. [10] For a fixed value of \( k \), define \( f(n) = \binom{n}{k} \). Prove that \( f(n) = O(n^k) \).

9. [10] Assuming \( x \) and \( y \) are integer variables, prove correct with respect to precondition “\( x \geq 0 \) and \( y \) is defined” and postcondition “\( x + y \neq 11 \)”:

\[
\text{if } y > 2 \text{ then} \quad \begin{align*}
x &:= y+6 \\
&\quad \text{if } x < 10 \text{ then} \quad y := 1 \\
&\quad \text{endif}
\end{align*}
\]
\[
\text{else} \quad \begin{align*}
x &:= y+4
\end{align*}
\]
\text{endif}

10. a. [10] Prove the following code is partially correct with respect to precondition “\( n \geq 0 \)” and postcondition “\( s = \sum_{i=1}^{n} a_i b_i \)” (assume \( k \) and \( s \) are integer variables and \( a \) and \( b \) are integer arrays of length at least \( n \)).:

\[
\begin{align*}
k &:= 1 \\
s &:= 0 \\
\text{while } k \leq n \text{ do} \\
&\quad s := s + (a[k] \times b[k]) \\
&\quad k := k+1
\end{align*}
\]

Be explicit about your loop invariant.

...b. [5] Prove that the loop terminates.
11. a. [10] Determine the weakest precondition with respect to the postcondition “\( w > 0 \)” for the following (assume \( w, z, y, \) and \( x \) are integer variables and that \( y \) and \( z \) are defined):

\[
\begin{align*}
x &:= y \\
y &:= x \\
x &:= z \\
y &:= x \\
w &:= x+y+z
\end{align*}
\]

b. [5] For the same piece of code, determine the weakest precondition with respect to the postcondition “\( wy = 12 \)”

12. [10] Determine the weakest precondition with respect to the postcondition “\( y = 1 \)” for the following code (assume \( z, y, \) and \( x \) are integer variables and that \( x \) and \( z \) are defined):

\[
\begin{align*}
\text{if } x < 3 \text{ then} \\
& \\
& y := z \\
& \quad \text{if } y < z \text{ then} \\
& \quad \\
& \quad y := 2^y \\
& \quad \text{endif} \\
& \text{else} \\
& \quad y := z-y \\
& \text{endif}
\end{align*}
\]