Asymptotic Dominance Problems

1. Display a function $f: N \to R$ that is $\mathrm{O}(1)$ but is not constant.

The function $f(n) = \begin{cases} 0 & if\ n = 0 \\ 1 & if\ n > 0 \end{cases}$ is not constant but for $n \geq 0, \quad |f(n)| \leq 1 \cdot |1|$.

2. Define the relation "$\leq$" on functions from $\boldsymbol{N}$ into $\boldsymbol{R}$ by $f \leq g$ if and only if $f = \mathrm{O}(g)$. Prove that $\leq$ is reflexive and transitive. (Recall: to be *reflexive*, you must

To prove reflexivity, notice that for any $f: N \to R$ and all $n \geq 0$, $|f(n)| \leq 1 \cdot |f(n)|$.

To prove transitivity, suppose $f = \mathrm{O}(g)$ and $g = \mathrm{O}(h)$, then by definition, there exist $N_f \geq 0,\ M_f \geq 0,\ N_g \geq 0,\ M_g \geq 0,$ so that for $n \geq N_f$, $|f(n)| \leq M_f |g(n)|$ and for $n \geq N_g$, $|g(n)| \leq M_g |h(n)|$. Thus for $n \geq \max\{N_f, N_g\}$, $|f(n)| \leq M_f M_g |h(n)|$. We may conclude that $f = \mathrm{O}(h)$.

3. Suppose $f = \mathrm{O}(g)$ and $g = \mathrm{O}(h)$, prove or disprove (with a simple counter-example) that $f = \mathrm{O}(h)$.

Suppose $f = \mathrm{O}(g)$ and $g = \mathrm{O}(h)$, then by definition, there exist $N_f \geq 0,\ M_f \geq 0,\ N_g \geq 0,\ M_g \geq 0,$ so that for $n \geq N_f$, $|f(n)| \leq M_f |g(n)|$ and for $n \geq N_g$, $|g(n)| \leq M_g |h(n)|$. Thus for $n \geq \max\{N_f, N_g\}$, $|f(n)| \leq M_f M_g |h(n)|$. We may conclude that $f = \mathrm{O}(h)$.

4. Suppose $f = o(g)$ and $g = \mathrm{O}(h)$. Prove that $f = o(h)$.

Since $g = \mathrm{O}(h),$ there exist M_1 and N_1 so that $n \geq N_1 \Rightarrow |\, g(n) \,| \leq M_1 \,|\, h(n) \,|$. Given $\varepsilon > 0,$ let $\varepsilon' = \varepsilon / M_1$. Since $f = o(g)$, there exist N_2 such that $n \geq N_2 \Rightarrow |\, f(n) \,| \leq \varepsilon' \,|\, g(n) \,| = \varepsilon / M_1 \,|\, g(n) \,|$. Thus letting $N = \max\{N_1, N_2\}$, for $n \geq N$ we have $|\, f(n) \,| \leq \varepsilon / M \,|\, g(n) \,| \leq \varepsilon \,|\, h(n) \,|$ so $f = o(h)$.

5. Suppose $f = \mathrm{O}(g)$ and $g = \mathrm{O}(h)$. If $h = \mathrm{O}(f)$, prove that $h = \mathrm{O}(g)$.

By definition, there exist $N_f \geq 0,\ M_f \geq 0,\ N_h \geq 0,\ M_h \geq 0,$ so that for $n \geq N_f$, $|f(n)| \leq M_f |g(n)|$ and for $n \geq N_h$, $|h(n)| \leq M_h |f(n)|$. Thus for $n \geq \max\{N_f, N_h\}$, $|h(n)| \leq M_f M_h |g(n)|$. We may conclude that $h = \mathrm{O}(g)$.

6. Using Theorem 2 and induction prove that if for $i = 1, 2, \ldots, k$, $f_i = O(g_i)$, then $\sum_{i=1}^{k} f_i = O(\sum_{i=1}^{k} |g_i|)$.

For k=1, we have $\sum_{i=1}^{1} f_i = f_1 = O(g_1) = O(\sum_{i=1}^{1} g_i)$. Now assume $\sum_{i=1}^{k} f_i = O(\sum_{i=1}^{k} |g_i|)$ and consider $\sum_{i=1}^{k+1} f_i$. Since $\sum_{i=1}^{k} f_i = O(\sum_{i=1}^{k} |g_i|)$ and $f_{k+1} = O(g_{k+1})$, Theorem 2 guarantees that $\sum_{i=1}^{k+1} f_i = \sum_{i=1}^{k} f_i + f_{k+1} = O(\sum_{i=1}^{k} |g_i| + |g_{k+1}|) = O(\sum_{i=1}^{k+1} |g_i|)$.

7. Employing induction and Theorem 3, prove that if for $i = 1, 2, \ldots, k$, $f_i = O(g)$, then $\sum_{i=1}^{k} f_i = O(g)$.

For k=1, we have $\sum_{i=1}^{1} f_i = f_1 = O(g)$ by hypothesis. Now assume $\sum_{i=1}^{k} f_i = O(g)$ and consider $\sum_{i=1}^{k+1} f_i$. Since $\sum_{i=1}^{k} f_i = O(g)$ and $f_{k+1} = O(g)$, Theorem 3 guarantees that $\sum_{i=1}^{k+1} f_i = \sum_{i=1}^{k} f_i + f_{k+1} = O(\max\{|g|, |g|\}) = O(|g|) = O(g)$.

8. Show that if $f(n) = 12n + 3$ and $g(n) = n^2$, then $f = O(g)$.

Let $N = 3$ and $M = 13$. For $n \geq N$:

$$|f(n)| = |12n + 3| = 12n + 3 \leq 12n + n = 13n \leq 13n^2 = 13|n^2| = M|g(n)|.$$

Thus $f = O(g)$.

9. Define $f: N \to R$ by $f(n) = \begin{cases} 10^{100} & \textit{for } n = 17 \\ n & \textit{for } n \neq 17 \end{cases}$. Prove that $f = O(n)$.

For $n \geq 18$, $|f(n)| = |n| \leq 1 \cdot |n|$, so $f = O(n)$.

10. Consider the functions f and g defined on $\mathbf{N}$ by $f(n) = \begin{cases} n^2 & \text{for } n \text{ even} \\ 2n & \text{for } n \text{ odd} \end{cases}$ and $g(n) = n^2$. Show that $f = \mathrm{O}(g)$ but that $f \neq o(g)$ and $g \neq \mathrm{O}(f)$.

$f = \mathrm{O}(g)$: Since for $n \geq 0$, $2n \leq 2n^2$; we have that $|2n| \leq 2|n^2|$ and $|n^2| \leq 2|n^2|$, so $|f(n)| \leq 2|g(n)|$. Thus $f = \mathrm{O}(g)$.

$f \neq o(g)$: Suppose $f = o(g)$, then for $\varepsilon = 1/2$ there is a non-negative N so that for all $n \geq N$, $|f(n)| \leq \varepsilon |g(n)|$. But letting $n = 2$ if $N = 0$ and $n = N$ or $N+1$ (whichever is even) if N is positive, we have $|f(n)| = n^2 > \frac{1}{2}n^2 = \varepsilon |g(n)|$. This is a contradiction, so $f \neq o(g)$

$g \neq \mathrm{O}(f)$.: Suppose $g = \mathrm{O}(f)$., then there exist nonnegative M and N so that for all $n \geq N$, $|g(n)| \leq M|f(n)|$. But letting n be odd and greater than N and $2M$, then we have $|g(n)| = n^2 = n \cdot n > 2Mn = M|2n| = M|f(n)|$. This is a contradiction, so $g \neq \mathrm{O}(f)$.

11. Show that $2^n = \mathrm{O}(n!)$.

For $n \geq 2$ and $i = 2, 3, \ldots, n$, we have $2 \leq i$, thus $\prod_{i=2}^{n} 2 \leq \prod_{i=2}^{n} i$. Therefore, $2^n = \prod_{i=1}^{n} 2 = 2 \cdot \prod_{i=2}^{n} 2 \leq 2 \cdot \prod_{i=2}^{n} i = 2 \cdot \prod_{i=1}^{n} i = 2\, n!$ and we have $|2^n| \leq 2 \cdot |n!|$, thus $2^n = \mathrm{O}(n!)$.

12. Show that for any real value of a, $a^n = \mathrm{O}(n!)$. (Hint: be careful to consider negative values of a.)

Define $K = \lceil |a| \rceil$ (i.e. K is the first integer greater than or equal to $|a|$). For $n \geq K$ and $i = K, K+1, \ldots, n$, we have $|a| \leq i$, thus $\prod_{i=K}^{n} |a| \leq \prod_{i=K}^{n} i$. Therefore, $|a|^n = \prod_{i=1}^{n} |a| = |a|^{K-1} \cdot \prod_{i=K}^{n} |a| \leq |a|^{K-1} \cdot \prod_{i=K}^{n} i \leq |a|^{K-1} \cdot \prod_{i=1}^{n} i = |a|^{K-1}\, n!$. So with $M = |a|^{K-1}$ and $N = K$, we have $|a^n| \leq M \cdot |n!|$ for all $n \geq N$. Thus $a^n = \mathrm{O}(n!)$.

13. Show that for any $b > 1$, $\log_b n = o(n)$

Consider any positive ε, and choose $N = \left\lceil 1 + \frac{2}{(b^\varepsilon - 1)^2} \right\rceil$. Then, if $n > N$, we have $n > 1 + \frac{2}{(b^\varepsilon - 1)^2}$, thus $\frac{(n-1)}{2}(b^\varepsilon - 1)^2 > 1$, and $\frac{n(n-1)}{2}(b^\varepsilon - 1)^2 > n$. But using the binomial theorem, we have

$$b^{\varepsilon n} = (b^\varepsilon)^n = (1 + (b^\varepsilon - 1))^n = \sum_{j=0}^{n} \binom{n}{j} (b^\varepsilon - 1)^j > \binom{n}{2} (b^\varepsilon - 1)^2 > n.$$

By taking base b logarithms, we have

$$\varepsilon |n| = \varepsilon n = \log_b b^{\varepsilon n} > \log_b n = |\log_b n|.$$

14. Prove that if $0 \le a < b$, then $a^n = o(b^n)$

If $a = 0$, then for all $\varepsilon > 0$ and all $n \ge 1$, we have $|\ a^n\ | = 0 \le \varepsilon |\ b^n\ |$. Assume now that $a > 0$. Take $N = \ln(\varepsilon) / \ln(a/b)$ and (assuming $\varepsilon < 1$), for $n \ge N$, $n \cdot \ln(a/b) \le \ln(\varepsilon)$ and $|\ a^n\ | = a^n \le \varepsilon \cdot b^n = \varepsilon |\ b^n\ |$. (If $\varepsilon \ge 1$ then $|\ a^n\ | = a^n \le \cdot b^n \le \varepsilon \cdot b^n = \varepsilon |\ b^n\ |$ for $n \ge 0$.) Thus $a^n = o(b^n)$.

15. Prove that if $0 \le a < b$, then $n^a = o(n^b)$

Given any $\varepsilon > 0$, let $N = (1/\varepsilon)^{1/(b-a)}$. Notice then for $n \ge N = (1/\varepsilon)^{1/(b-a)}$, $n^{b-a} \ge 1/\varepsilon$, and $n^{-(b-a)} \le \varepsilon$. So $|n^a| = |n^{-(b-a)}\ n^b| = |n^{-(b-a)}|\,|n^b| \le \varepsilon\ |n^b|$. Therefore, $n^a = o(n^b)$.

16. Prove that if $0 < a < b$, then $b^n \ne O(a^n)$.

Given $M \ge 0$ and $N \ge 0$, let $\overline{M} = \max\{M, 1\}$ thus $\overline{M} \ge M$ and $\ln(\overline{M}) \ge 0$. Notice that $\ln(\frac{b}{a}) > 0$ and choose $n = \max\{N, \left\lceil \frac{\ln(\overline{M})}{\ln(\frac{b}{a})} \right\rceil\} + 1$. For this n we have $n > \frac{\ln(\overline{M})}{\ln(\frac{b}{a})}$, thus $n \ln(\frac{b}{a}) > \ln(\overline{M})$ and $(\frac{b}{a})^n > \overline{M} \ge M$. But then $|b^n| = b^n > M\ a^n = M\ |a^n|$ so $b^n \ne O(a^n)$

17. Prove that $\sqrt{n} = O(n^2)$.

Let $M = 1$ and $N = 1$. For $n \ge N, n^{3/2} \ge 1$. Thus $|\sqrt{n}| = \sqrt{n} \le n^{3/2}\sqrt{n} = n^2 = 1|n^2|$, so $\sqrt{n} = O(n^2)$.

18. Prove that $e^{(n^2)} \ne o(e^n)$.

Let $\varepsilon = 1$, consider and N, and choose $n \ge \max\{N,2\}$. Since $n \ge 2$, $n^2 \ge 2n > n$ and $|e^{n^2}| > e^n = \varepsilon |n|$ so $e^{(n^2)} \ne o(e^n)$.

19. Using only Definition 1, prove that $3n^4 = O(n^{4.5})$.

Let $M = 3$ and $N = 1$. For $n \ge N = 1$, we have $\sqrt{n} \ge 1$, so $|3n^4| \le 3\ n^4\sqrt{n} = 3|n^{4.5}|$. Thus $3n^4 = O(n^{4.5})$.

20. Using only Definition 2, prove that $5^n \ne o(2 \cdot 4^n)$.

Let $\varepsilon = 1/4$ and suppose there exists N so that for all $n \ge N$, $|5^n| \le \varepsilon |2 \cdot 4^n|$. But for $n = \max\{1, \lceil N \rceil\}$, we have $n \ge N$ and $n \ge 1$, so $(\frac{5}{4})^n > 1$ and $5^n > 4^n$, thus $|5^n| = 5^n > 4^n = 1/2|2 \cdot 4^n| = \varepsilon |2 \cdot 4^n|$ and $5^n \ne o(2 \cdot 4^n)$.

21. Show that if $f(n) = n^2$ and $g(n) = n$, then $f \ne o(g)$.

Let $\varepsilon = 1$ and consider any positive N. Let $n = N + 1$ so $n \ge 2$ and $n \ge N$. We have:

$$|f(n)| = |n^2| = |n| \cdot |n| \ge 2|n| > \varepsilon |n| = \varepsilon |g(n)|.$$

Thus $f \ne o(g)$.

22. Show that $\log_2 n! = O(n \log_2 n)$ and $n \log_2 n = O(\log_2 n!)$.

For $n \ge 1$, we have $\log_2 n! = \log_2(\prod_{i=1}^{n} i) = \sum_{i=1}^{n} \log_2 i \le \sum_{i=1}^{n} \log_2 n = n \log_2 n$. Thus $|\log_2 n!| \le 1 \cdot |n \log n|$ and $\log_2 n! = O(n \log_2 n)$. To show $n \log_2 n = O(\log_2 n!)$ let $N = 8$ and $M = 3$. Notice that if $n \ge 8$, $\frac{n}{8} \ge 1$, so $(\frac{n}{2})^3 = \frac{n}{8} n^2 \ge n^2$. Also notice that $\left\lceil \frac{n}{2} \right\rceil - 1 \le \frac{n}{2}$, so $n - \left\lceil \frac{n}{2} \right\rceil + 1 \ge n - \frac{n}{2} = \frac{n}{2}$. Finally

$$n^n = (n^2)^{n/2} \le ((\frac{n}{2})^3)^{n/2} = (\frac{n}{2})^{3n/2} \le (\frac{n}{2})^{3(n - \left\lceil \frac{n}{2} \right\rceil + 1)} = \prod_{k=\left\lceil \frac{n}{2} \right\rceil}^{n} \left\lceil \frac{n}{2} \right\rceil^3 \le \prod_{k=\left\lceil \frac{n}{2} \right\rceil}^{n} k^3 \le \prod_{k=1}^{n} k^3 = (n!)^3.$$

By taking logs, we have for $n \geq 8$, $|n \log_2 n| = n \log_2 n \leq 3 \log_2 n! = 3 |\log_2 n!|$.