Introductory Combinatorial Theory

1. Rule of Sum: If to make a selection there are *m* options of one type and *n* distinct options of another type, then there are *m+n* total options if only one or the other option is taken.

$$|A \cup B| = |A| + |B| \text{ if } A \cap B = \varnothing.$$

2. Rule of Product: If to make a selection there are *m* options of one type and *n* options of another type, then there are *mn* total options if both one and the other option is taken.

$$|A \times B| = |A| \cdot |B|.$$

3. Ordered sampling with repetition not allowed: If there are *n* choices for each element of an array of length *m* but no choice can be repeated, then $n \geq m$ and there are *P(n, m)* = $\dfrac{n!}{(n-m)!}$ such arrays.

The cardinality of the set of one-to-one functions from B to A is P(|A|, |B|)

4. Ordered sampling with repetition allowed: If there are *n* independent choices for each element of an array of length *m*, there are n^m such arrays.

The cardinality of the set of functions from B to A is $|A|^{|B|}$

3. Unordered sampling with repetition not allowed: If there are *n* choices for each element of an array of length *m* but no choice can be repeated and the order of the elements in the array is ignored, then $n \geq m$ and there are *C(n, m)* = $\dfrac{n!}{m!(n-m)!} = \dbinom{n}{m}$ such arrays.

There are C(n, m) subsets of cardinality m from a set of cardinality n.

6. Unordered sampling with repetition allowed: If there are *n* independent choices for each element of an array of length *m* and the order of the elements in the array is ignored, there are *C(n+m-1, m)* $= \dbinom{n+m-1}{m}$ such arrays

There are C(n+m-1, m) multisubsets of cardinality m from a set of cardinality n.

Theorem: Let $A_1, A_2, \ldots, A_n$ be finite sets. The cardinality of their union satisfies

$$\#(\bigcup_{i=1}^{n} A_i) =$$

$$\sum_{1 \leq i_1 \leq n} \#(A_{i_1}) - \sum_{1 \leq i_1 < i_2 \leq n_1} \#(A_{i_1} \cap A_{i_2}) + \sum_{1 \leq i_1 < i_2 < i_3 \leq n_1} \#(A_{i_1} \cap A_{i_2} \cap A_{i_3}) - + \cdots + (-1)^{n-1} \#(A_1 \cap A_2 \cap \ldots \cap A_n)$$

Introductory Combinatorial Probability Theory

1. If an event has *m* equally likely outcomes and *n* of those are favorable, the probability of the favorable event is n/m. If the favorable event - the set of favorable outcomes - is denoted by *E*, the probability is *Pr(E).*

2. Conditional Probability: The probability of the event *A* given the event *B* is $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$.

3. Independence: If $\Pr(A|B) = \Pr(A)$, then event *A* is "statistically independent" of event *B*. Equivalently, $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$.

Theorem: Let $A_1, A_2, \ldots, A_n$ be events. The probability of the union of the events satisfies

$$\Pr(\bigcup_{i=1}^{n} A_i) =$$

$$\sum_{1 \le i_1 \le n} \Pr(A_{i_1}) - \sum_{1 \le i_1 < i_2 \le n_1} \Pr(A_{i_1} \cap A_{i_2}) + \sum_{1 \le i_1 < i_2 < i_3 \le n_1} \Pr(A_{i_1} \cap A_{i_2} \cap A_{i_3}) - + \cdots + (-1)^{n-1} \Pr(A_1 \cap A_2 \cap \ldots \cap A_n)$$