Introductory Combinatorial Theory

1. **Rule of Sum**: If to make a selection there are \(m\) options of one type and \(n\) distinct options of another type, then there are \(m+n\) total options if only one or the other option is taken.

\[
|A \cup B| = |A| + |B| \text{ if } A \cap B = \emptyset.
\]

2. **Rule of Product**: If to make a selection there are \(m\) options of one type and \(n\) options of another type, then there are \(mn\) total options if both one and the other option is taken.

\[
|A \times B| = |A||B|.
\]

3. **Ordered sampling with repetition not allowed**: If there are \(n\) choices for each element of an array of length \(m\) but no choice can be repeated, then \(n \geq m\) and there are \(P(n, m) = \frac{n!}{(n-m)!}\) such arrays.

*The cardinality of the set of one-to-one functions from \(B\) to \(A\) is \(P(|A|, |B|)\)*

4. **Ordered sampling with repetition allowed**: If there are \(n\) independent choices for each element of an array of length \(m\), there are \(n^m\) such arrays.

*The cardinality of the set of functions from \(B\) to \(A\) is \(A^{[n]}\)*

5. **Unordered sampling with repetition not allowed**: If there are \(n\) choices for each element of an array of length \(m\) but no choice can be repeated and the order of the elements in the array is ignored, then \(n \geq m\) and there are \(C(n, m) = \frac{n!}{m!(n-m)!} = \binom{n}{m}\) such arrays.

*There are \(C(n, m)\) subsets of cardinality \(m\) from a set of cardinality \(n\).*

6. **Unordered sampling with repetition allowed**: If there are \(n\) independent choices for each element of an array of length \(m\) and the order of the elements in the array is ignored, there are \(C(n+m-1, m) = \binom{n+m-1}{m}\) such arrays

*There are \(C(n+m-1, m)\) multisubsets of cardinality \(m\) from a set of cardinality \(n\).*

**Theorem**: Let \(A_1, A_2, \ldots, A_n\) be finite sets. The cardinality of their union satisfies

\[
\#(\bigcup_{i=1}^{n} A_i) = \sum_{I \subseteq \{1, \ldots, n\}} \#(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) - \sum_{I \subseteq \{1, \ldots, n\}} \#(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_{\ell+1}}) + \cdots + (-1)^{n-1} \#(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_n)
\]
1. If an event has \( m \) equally likely outcomes and \( n \) of those are favorable, the probability of the favorable event is \( \frac{n}{m} \). If the favorable event - the set of favorable outcomes - is denoted by \( E \), the probability is \( Pr(E) \).

2. **Conditional Probability**: The probability of the event \( A \) given the event \( B \) is
   \[
   Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}.
   \]

3. **Independence**: If \( Pr(A|B) = Pr(A) \), then event \( A \) is "statistically independent" of event \( B \). Equivalently, \( Pr(A \cap B) = Pr(A) \cdot Pr(B) \).

**Theorem**: Let \( A_1, A_2, \ldots, A_n \) be events. The probability of the union of the events satisfies

\[
Pr\left( \bigcup_{i=1}^{n} A_i \right) = \\
\sum_{I \subseteq [n]} \Pr(A_I) - \sum_{I \subseteq [n]} \Pr(A_{I_1} \cap A_{I_2}) + \sum_{I \subseteq [n]} \Pr(A_{I_1} \cap A_{I_2} \cap A_{I_3}) - \cdots + (-1)^{n-1} \Pr(A_{I_1} \cap A_{I_2} \cap \cdots \cap A_n)
\]