Homework 4 Solutions
CS 336

The important issue is the logic you used to arrive at your answer.

1. In how many ways can you partition a set of $2n$ elements into a collection of $n$ pairs?

Imagine placing the $2n$ elements in a row from left to right. For the leftmost element, there are $2n-1$ possibilities for its pair-mate. Once these two elements have been removed, there are $2n-2$ elements remaining. For the one that is now leftmost, there are $2n-3$ choices for its pair-mate. This continues until there is a single pair remaining. Thus, the total number of choices is $(2n-1)(2n-3)(2n-5)\cdots5\cdot3\cdot1$.
(Sometimes this is denoted by $(2n-1)!!$.)

2. How many $n$-digit numbers have their digits in non-decreasing order? (i.e. for $n = 6$, the number 344455 is counted but 123465 is not - and 012345 is not a 6 digit number.)

Recognize that an $n$-digit number whose digits are in non-decreasing order is totally characterized by the positions where the digits change. That is, if we know where the transition is from 1 to 2, from 2 to 3, ..., and from 8 to 9, then we know the number. (For example, if $n=3$ and we are told that the transition from 1 to 2 occurs prior to the first digit, that the transitions from 2 to 3, 3 to 4, ..., 7 to 8 occur between the first and second positions, and that the transition from 8 to 9 occurs after the last position, we know the number must be 288.) There are 8 such transitions and they may occur prior to any of the $n$ digit positions as well as after the final position. Thus there are $n+1$ such locations where the 8 transitions may occur. This is equivalent to choosing 8 from $n+1$ with replacement and thus the number is $\binom{(n+1)+8-1}{8} = \binom{n+8}{8}$. 

3. Use a combinatorial argument to prove:

\[ \binom{2n+2}{n+1} = \binom{2n}{n+1} + 2 \binom{2n}{n} + \binom{2n}{n-1}. \]

(Hint: Clearly from the left hand side you will be choosing subsets of size \( n+1 \) from a set of \( 2n+2 \) elements. Consider a way of decomposing such choosing options into several disjoint cases.)

Consider a set \( A \) of \( 2n \) elements together with distinct elements \( b \) and \( c \), neither of which are elements of \( A \). Letting \( D = A \cup \{b\} \cup \{c\} \), we notice that \( D \) has cardinality \( 2n+2 \) and thus has \( \binom{2n+2}{n+1} \) subsets of size \( n+1 \). Alternatively, we could count these subsets by recognizing that each subset is either

1. A set of \( n+1 \) elements entirely from \( A \),
2. The element \( \{b\} \) and a set of \( n \) elements from \( A \),
3. The element \( \{c\} \) and a set of \( n \) elements from \( A \),
or
4. Both elements \( \{b\} \) and \( \{c\} \) and \( n-1 \) elements from \( A \),

and that no such subset is in two or more of these cases. The total count for all subsets of size \( n+1 \) is then \( \binom{2n}{n+1} + 2 \binom{2n}{n} + \binom{2n}{n-1} \).