Homework 4 Solutions
CS 336

The important issue is the logic you used to arrive at your answer.

1. In how many ways can you partition a set of 2*n* elements into a collection of *n* pairs?

Imagine placing the 2*n* elements in a row from left to right. For the leftmost element, there are 2*n*-1 possibilities for its pair-mate. Once these two elements have been removed, there are 2*n*-2 elements remaining. For the one that is now leftmost, there are 2*n*-3 choices for its pair-mate. This continues until there is a single pair remaining. Thus, the total number of choices is $(2n-1)(2n-3)(2n-5)\cdots 5\cdot 3\cdot 1$.
(Sometimes this is denoted by (2*n*-1)!! .)

2. How many *n*-digit numbers have their digits in non-decreasing order? (i.e. for *n* = 6, the number 344455 is counted but 123465 is not - and 012345 is not a 6 digit number.)

Recognize that an *n*-digit number whose digits are in non-decreasing order is totally characterized by the positions where the digits change. That is, if we know where the transition is from 1 to 2, from 2 to 3, ..., and from 8 to 9, then we know the number. (For example, if *n*=3 and we are told that the transition from 1 to 2 occurs prior to the first digit, that the transitions from 2 to 3, 3 to 4, ..., 7 to 8 occur between the first and second positions, and that the transition from 8 to 9 occurs after the last position, we know the number must be 288.) There are 8 such transitions and they may occur prior to any of the *n* digit positions as well as after the final position. Thus there are *n*+1 such locations where the 8 transitions may occur. This is equivalent to choosing 8 from *n*+1 with replacement and thus the number is $\binom{(n+1)+8-1}{8} = \binom{n+8}{8}$.

3. Use a combinatorial argument to prove:

$$\binom{2n+2}{n+1} = \binom{2n}{n+1} + 2\binom{2n}{n} + \binom{2n}{n-1}.$$

(Hint: Clearly from the left hand side you will be choosing subsets of size n+1 from a set of 2n+2 elements. Consider a way of decomposing such choosing options into several disjoint cases.)

Consider a set A of 2n elements together with distinct elements b and c, neither of which are elements of A. Letting $D = A \cup \{b\} \cup \{c\}$, we notice that D has cardinality 2n+2 and thus has $\binom{2n+2}{n+1}$ subsets of size n+1. Alternatively, we could count these subsets by recognizing that each subset is either

1. A set of n+1 elements entirely from A,
2. The element $\{b\}$ and a set of n elements from A,
3. The element $\{c\}$ and a set of n elements from A,

or 4. Both elements $\{b\}$ and $\{c\}$ and n-1 elements from A,

and that no such subset is in two or more of these cases. The total count for all subsets of size n+1 is then $\binom{2n}{n+1} + 2\binom{2n}{n} + \binom{2n}{n-1}$.