Homework 22 Solutions
CS 336

The important issue is the logic you used to arrive at your answer.

1. How many bit strings contain exactly eight 0s and ten 1s if every 0 must be immediately followed by a 1?

Equivalently, the bit strings must consist of eight 01 substrings and two 1s. Thus, there are ten total positions and choosing the two positions for the 1s determines the string. There are $\binom{10}{2}$ such strings.

2. a. Let A be a set of cardinality m and A^k be its k-fold Cartesian product with itself. Let B be a set of cardinality n. How many different functions map from A^k to B?

Since the cardinality of A^k is m^k and a function has n independent options for each of these elements, there are $n^{(m^k)}$ such functions.

b. How many Boolean (i.e. truth- or false - valued) functions are defined for pairs of Boolean variables? (Hint: What are m, n, and k here?)

Here we have $m = 2$, $k = 2$, and $n = 2$ so there are $2^4 = 16$ functions.

3. Present a combinatorial argument that for all positive values of m, n, and r:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Consider distinct sets A and B of cardinalities m and n, respectively. There are $\binom{m+n}{r}$ subset of $A \cup B$ of size r. Alternatively, for any such subset, there must be some r-k elements of A and k elements of B for a value of k between 0 and r. For a fixed k there are $\binom{m}{r-k}\binom{n}{k}$ such subsets and thus $\sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$ overall.

4. Consider a three-dimensional array with dimensions $m \times n \times p$. Now consider a path from the (1, 1, 1) element to the (m, n, p) element such that at each step exactly one of the indices increases by exactly one (i.e., from any (i, j, k), a step is taken to either (i+1, j, k), (i, j+1, k) or (i, j, k+1)). How many such paths are there?

There are $m+n+p-3$ steps in the path and any path is determined by the positions in the sequence at which the first index increases and at which the second index increases. These increases occur $m-1$ and $n-1$ times, respectively. Thus there are $\begin{pmatrix} m+n+p-3 \\ m-1 \quad n-1 \end{pmatrix}$ such paths.

5. Prove by induction that for $n \geq 1$ and any set of events $E_1, E_2, ..., E_n$ for which probabilities are defined:

$$\Pr(\bigcup_{i=1}^{n} E_i) \leq \sum_{i=1}^{n} \Pr(E_i)$$

For $n = 1$, we have $\Pr(\bigcup_{i=1}^{n} E_i) = \Pr(E_1) = \sum_{i=1}^{n} \Pr(E_i)$. Assuming the result is true for a given $n \geq 1$, we have

$$\begin{aligned} \Pr(\bigcup_{i=1}^{n+1} E_i) = \Pr(\bigcup_{i=1}^{n} E_i \cup E_{n+1}) &= \Pr(\bigcup_{i=1}^{n} E_i) + \Pr(E_{n+1}) - \Pr(\bigcup_{i=1}^{n} E_i \cap E_{n+1}) \\ &\leq \Pr(\bigcup_{i=1}^{n} E_i) + \Pr(E_{n+1}) = \sum_{i=1}^{n+1} \Pr(E_i) \end{aligned}$$