Homework 23 Solutions
CS 336

The important issue is the logic you used to arrive at your answer.

1. Let A be the set of finite length strings of 0s and 1s and let $\boldsymbol{s}$ = <010101...> (i.e., the infinite string alternating 0s and 1s). Let B be the set of all infinite strings of the form $\boldsymbol{a}||\boldsymbol{s}$ where $\boldsymbol{a} \in A$ and || indicates concatenation. Is B finite, countably infinite, or uncountably infinite? Prove your assertion.

For $n \geq 0$, the set A_n of bit strings of length n is finite. The set B_n composed of strings from A_n concatenated with $\boldsymbol{s}$ is also finite. Since $B = \bigcup_{n=0}^{\infty} B_n$, B is the countable union of a set of countable sets. Therefore, B is countable. Since B contains the infinite subset {$\boldsymbol{s}$, 0||$\boldsymbol{s}$, 00||$\boldsymbol{s}$, 000||$\boldsymbol{s}$, ...} it cannot be finite, thus B is countably infinite.

2. Using only Definition 2', show that the unit circle $A = \{(x,y) | x \text{ and } y \text{ are real and } x^2 + y^2 \leq 1\}$ is infinite.

Define $f: A \to A$ by $f(x,y) = (x/2, y/2)$. Since $(x_1, y_1) \neq (x_2, y_2)$ implies that $(x_1/2, y_1/2) \neq (x_2/2, y_2/2)$, f is one-to-one. Furthermore, since $(x/2)^2 + (y/2)^2 = (x^2 + y^2)/4 \leq 1/4 < 1$, f maps A into a proper subset of itself.

3. Consider arrays of positive integers whose sum is 17 (e.g., <17>, <9, 8>, and <1, 5, 1, 6, 4>). Is the set of all such arrays finite, countably infinite, or uncountably infinite? Prove your assertion.

We shall prove this set is finite by showing that it is a subset of a finite set. Consider the set of arrays of length l for $1 \leq l \leq 17$ whose integer elements must have values between 1 and 17. This set is finite and the collection of all such sets for $1 \leq l \leq 17$ is also finite. The set of positive integer arrays with sum 17 is a subset of this, thus must be finite as well.

4. Since there are $n!$ permutations of n elements, a binary decision tree whose leaves are the $n!$ permutations must have height $\lceil \log_2 n! \rceil$. This is thus the number of comparisons necessary to determine which permutation one has - thus to sort the elements. Yet this number is often quoted as $n \log_2 n$. Prove that $\log_2 n! = O(n \log_2 n)$.

For $n \geq 1$, we have $\log_2 n! = \log_2 (\prod_{i=1}^{n} i) = \sum_{i=1}^{n} \log_2 i \leq \sum_{i=1}^{n} \log_2 n = n \log_2 n$. Thus $|\log_2 n!| \leq 1 \cdot |n \log n|$ and $\log_2 n! = O(n \log_2 n)$.

5. Show with a simple counter-example that even if $f = O(g)$ and $a > 1$, it does not follow that $a^f = O(a^g)$. (Hint: you may want to use that if $0 < c < b$, then $b^n \neq O(c^n)$.)

Let $a = 2$, and for $n \geq 0$, $f(n) = n$ and $g(n) = n/2$. Clearly $f = O(g)$, since for $n \geq 0, |n| \leq 2 \cdot |n/2|$. However, $a^{g(n)} = 2^{n/2} = (\sqrt{2})^n$. Since $0 < \sqrt{2} < 2$, $2^n \neq O((\sqrt{2})^n)$ so $a^f \neq O(a^g)$.

6. Suppose $f_1 = O(g_1)$ and $f_2 = o(g_2)$, show $f_1 + f_2 = O(|g_1| + |g_2|)$.

By definition, there exist non-negative constants M_1 and N_1 such that for all $n \geq N_1$, $|f_1(n)| \leq M_1 |g_1(n)|$ and for $\varepsilon = M_1$, there exist non-negative constant N_2 such that for all $n \geq N_2$, $|f_2(n)| \leq \varepsilon |g_2(n)| = M_1 |g_2(n)|$. For $n \geq \max\{N_1, N_2\}$ both inequalities hold so:
$|f_1(n) + f_2(n)| \leq |f_1(n)| + |f_2(n)| \leq M_1 |g_1(n)| + M_1 |g_2(n)| \leq M_1 (|g_1(n)| + |g_2(n)|)$.
Therefore, $f_1 + f_2 = O(|g_1| + |g_2|)$.