Homework 23 Solutions
CS 336

The important issue is the logic you used to arrive at your answer.

1. Let $A$ be the set of finite length strings of 0s and 1s and let $s = <010101...>$ (i.e., the infinite string alternating 0s and 1s). Let $B$ be the set of all infinite strings of the form $a || s$ where $a \in A$ and $||$ indicates concatenation. Is $B$ finite, countably infinite, or uncountably infinite? Prove your assertion.

For $n \geq 0$, the set $A_n$ of bit strings of length $n$ is finite. The set $B_n$ composed of strings from $A_n$ concatenated with $s$ is also finite. Since $B = \bigcup_{n=0}^{\infty} B_n$, $B$ is the countable union of a set of countable sets. Therefore, $B$ is countable. Since $B$ contains the infinite subset $\{s, 0||s, 00||s, 000||s, \ldots\}$ it cannot be finite, thus $B$ is countably infinite.

2. Using only Definition 2', show that the unit circle $A = \{(x, y) | x and y are real and $x^2 + y^2 \leq 1$\}$ is infinite.

Define $f: A \rightarrow A$ by $f(x, y) = (x/2, y/2)$. Since $(x_1, y_1) \neq (x_2, y_2)$ implies that $(x_1/2, y_1/2) \neq (x_2/2, y_2/2)$, $f$ is one-to-one. Furthermore, since $(x/2)^2 + (y/2)^2 = (x^2 + y^2)/4 \leq 1/4 < 1$, $f$ maps $A$ into a proper subset of itself.

3. Consider arrays of positive integers whose sum is 17 (e.g., <17>, <9, 8>, and <1, 5, 1, 6, 4>). Is the set of all such arrays finite, countably infinite, or uncountably infinite? Prove your assertion.

We shall prove this set is finite by showing that it is a subset of a finite set. Consider the set of arrays of length $l$ for $1 \leq l \leq 17$ whose integer elements must have values between 1 and 17. This set is finite and the collection of all such sets for $1 \leq l \leq 17$ is also finite. The set of positive integer arrays with sum 17 is a subset of this, thus must be finite as well.
4. Since there are \( n! \) permutations of \( n \) elements, a binary decision tree whose leaves are the \( n! \) permutations must have height \( \lceil \log_2 n! \rceil \). This is thus the number of comparisons necessary to determine which permutation one has - thus to sort the elements. Yet this number is often quoted as \( n \log_2 n \). Prove that \( \log_2 n! = O(n \log_2 n) \).

For \( n \geq 1 \), we have \( \log_2 n! = \log_2 \left( \prod_{i=1}^{n} i \right) = \sum_{i=1}^{n} \log_2 i \leq \sum_{i=1}^{n} \log_2 n = n \log_2 n \). Thus \( |\log_2 n!| \leq 1 \log_2 n \) and \( \log_2 n! = O(n \log_2 n) \).

5. Show with a simple counter-example that even if \( f = O(g) \) and \( a > 1 \), it does not follow that \( a^f = O(a^g) \). (Hint: you may want to use that if \( 0 < c < b \), then \( b^n \neq O(c^n) \).)

Let \( a = 2 \), and for \( n \geq 0 \), \( f(n) = n \) and \( g(n) = n/2 \). Clearly \( f = O(g) \), since for \( n \geq 0 \), \( |n| \leq 2 |n/2| \). However, \( a^{f(n)} = 2^n = (\sqrt{2})^n \). Since \( 0 < \sqrt{2} < 2 \), \( 2^n \neq O((\sqrt{2})^n) \) so \( a^f \neq O(a^g) \).

6. Suppose \( f_1 = O(g_1) \) and \( f_2 = o(g_2) \), show \( f_1 + f_2 = O(|g_1| + |g_2|) \).

By definition, there exist non-negative constants \( M_1 \) and \( N_1 \) such that for all \( n \geq N_1 \), \( |f_1(n)| \leq M_1 |g_1(n)| \) and for \( \varepsilon = M_1 \), there exist non-negative constant \( N_2 \) such that for all \( n \geq N_2 \), \( |f_2(n)| \leq \varepsilon |g_2(n)| = M_1 |g_2(n)| \). For \( n \geq \max \{N_1, N_2\} \) both inequalities hold so:

\[
|f_1(n) + f_2(n)| \leq |f_1(n)| + |f_2(n)| \leq M_1 |g_1(n)| + M_1 |g_2(n)| \leq M_1 (|g_1(n)| + |g_2(n)|).
\]

Therefore, \( f_1 + f_2 = O(|g_1| + |g_2|) \).