

1. **[5]** Given sets A and B , each of cardinality $n \geq 1$, how many functions map A in a one-to-one fashion onto B ?

2. **[5]** a. Given the set of r symbols $\{a_1, a_2, \dots, a_r\}$, how many different strings of length $n \geq 1$ exist?

[5]b. Given the set of r symbols $\{a_1, a_2, \dots, a_r\}$, how many different strings of length $n \geq 2$ exist that contain at least one a_1 ?

3. **[10]** Present a combinatorial argument that for all positive integers x and y

$$\sum_{k=0}^n \binom{n}{k} x^k y^{n-k} = (x + y)^n.$$

(Hint: Consider sequences drawn from the union of distinct sets A and B of cardinalities x and y , respectively.)

4. **[5]** Given m a's, n b's, and p c's, how many distinct sequences are there that employ each of the $m + n + p$ symbols exactly once?

5. **[5]** If four six-sided dice are thrown, how many different configurations (ignoring order) are possible? (Understand: the configuration $\{2, 2, 3, 4\}$ is the same as the configuration $\{2, 3, 2, 4\}$ but different from $\{2, 3, 3, 4\}$)

6. **[10]** A pair of six-sided dice are thrown r times. (We ignore order in the pair but not in the sequence. Thus the sequence beginning $(1,2), (2,6), (1,3), \dots$ is the same as the sequence $(2,1), (2,6), (1,3), \dots$ but different from the sequence $(2,6), (1,2), (1,3), \dots$). How many such sequences have each of $(1,1), (2,2), \dots$, and $(6,6)$ appearing at least once within the sequence? (Hint: First figure how many configurations can be displayed in a single throw of the pair. Second, figure how many total sequences of length r there are, without considering the requirement for the 6 certain pairs. Lastly, let $A_i =$ the set of such sequences avoiding (i,i) and apply the inclusion/exclusion principle.)

7. **[5]** a. Suppose k numbers are drawn from $\{1, 2, \dots, n\}$ allowing repetitions. Considering order to be relevant, how many such strings are there?

[5] b. Now assume each of the different strings in part a. is equally likely. What is the probability that the maximum of the k numbers is greater than or equal to r , where $1 \leq r \leq n$?

8. **[5]** Consider the following tables of probabilities for getting certain grades in a course and being in the freshman class. What is the probability of getting an A given that a student is a freshman?

	Freshman	Non-Freshman
A	.3	.2
Less than A	.4	.1

9. **[10]** Suppose a message consists of 0's and 1's being transmitted with equal

likelihood until five 1's have been transmitted (i.e., the message terminates with the fifth 1). Give an expression for the expected value of the number of bits in the message? (Don't waste time trying to simplify the expression.)

1. [5] Given set A of cardinality $r \geq 1$ and set B , of cardinality $n \geq 1$, how many functions map A in a one-to-one fashion into B ?

2. [5] a. Given the set of r symbols $\{a_1, a_2, \dots, a_r\}$, how many different strings of length $n \geq 1$ exist?

[5]b. Given the set of r symbols $\{a_1, a_2, \dots, a_r\}$, how many different strings of length $n \geq 2$ exist that contain at least two a_1 's?

3. [10] Present a combinatorial argument that for all positive integers $1 \leq k \leq m \leq r$:

$$\binom{r}{m} \binom{m}{k} = \binom{r}{k} \binom{r-k}{m-k}.$$

4. [5] Given m a's, n b's, p c's and q d's, how many distinct sequences are there that employ each of the $m+n+p+q$ symbols exactly once?

5. Consider sequences of the form $\langle r_1, r_2, \dots, r_n \rangle$, where the $r_i \geq 0$. For fixed positive values of r and n , how many such sequences are there satisfying

$$r_1 + r_2 + \dots + r_n = r.$$

6. [5] a. Suppose $k \leq n$ and k numbers are drawn from $\{1, 2, \dots, n\}$ without allowing repetitions. Considering order to be relevant, how many such strings are there?

[5] b. Now assume each of the different strings in part a. is equally likely. What is the probability that the minimum of the k numbers is less than or equal to r , where $1 \leq r \leq n$?

7. [5] Consider the following tables of probabilities for getting certain scores on an exam and being a CS major or not. Is the event of getting at least a score of 80 independent of the event that the student is a CS major?

	<i>CS Major</i>	<i>non-CS Major</i>
90 to 100	.08	.02
80 to 89	.17	.06
70 to 79	.28	.10
60 to 69	.14	.08
0 to 59	.08	.02

8. [10] Suppose a message consists of a total of n 0's and 1's being transmitted with equal likelihood. Give an expression for the expected value of the number of 1's minus the number of 0's in the message? (Don't waste time trying to simplify the expression.)

1. [5] a. Suppose all people either have one word names (e.g. "Cher"), two word names (e.g., "Jack Sprat") or three word names (e.g. "Richard Milhous Nixon"). How many different sets of initials can people have?

[5]b. Suppose all 10 digit telephone numbers obeyed these rules:
 the first digit cannot be 1.
 The second digit must be 0 or 1
 the other 8 digits can be any decimal digit.
 How many such 10 digit telephone numbers are there?

2. [5] a. How many permutations of $a, b, c, d, e,$ and f have b to the left of c ?

[5]b. How many permutations of $a, b, c, d, e,$ and f have b to the left of c and e to the left of f ?

3. [5] a. Present a combinatorial argument that for all positive integers $n, a,$ and $b(>a)$:

$$\sum_{k=0}^n \binom{n}{k} a^k (b-a)^{n-k} = b^n .$$

[5] b. Present a combinatorial argument that for all positive integers n :

$$\binom{2n}{2} = 2\binom{n}{2} + n^2$$

4. [10] An alphabet A contains exactly n characters. A message consists of an ordered array of elements of A (permitting repetitions)? If for each position in the message each character is equally likely, what is the probability that a message of length n does not contain all n characters.

5. [10] Suppose m cards are to be drawn from a 52 card deck with repetition possible. What is the probability that cards 1, 2, ..., $m-1$ are not hearts but card m is a heart?

6. [10] Let E be a set of equally likely events and A and B be subsets of E . Show that

$$\Pr(A|B) = 1 + \Pr(A) / \Pr(B) - \Pr(A \cup B) / \Pr(B)$$

7. [5] Consider the following tables of probabilities for getting certain scores on an exam and being a CS major or not. Is the event of getting at least a score of 70 independent of the event that the student is a CS major?

	<i>CS Major</i>	<i>non-CS Major</i>
90 to 100	.08	.02
80 to 89	.17	.04
70 to 79	.28	.12
60 to 69	.14	.08
0 to 59	.08	.02

8. [10] Suppose a message consists of a total of n 0's and 1's being transmitted with equal likelihood. What is the expected number of 0's?

1. [10]. Given set A of cardinality $r \geq 1$ and set B , of cardinality $n \geq 1$, how many functions mapping A into B are **not** one-to-one?

2. [10] For $n \geq 2$, determine the number of strings of a 's and b 's of length n that either begin with the string ab or end with the string ba or both. (Do not ignore the cases of $n = 2$, and $n = 3$.)

3. a. [10] Assuming $1 \leq m, n$, Let $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$. How many strings using each of these symbols exactly once have the symbols of A and B occurring in the order given (i.e. in the string a_i must occur to the left of a_j if $1 \leq i < j \leq m$ and similarly for B)?

b. [10] How many strings using each of these symbols exactly once have the symbols of just A occurring in the order given but not necessarily B ?

4. [10] Using a combinatorial argument, prove that for $n \geq 1$:

$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$$

5. a. [5] For $n \geq 1$, assume all strings of length n from the set $\{a, b, c\}$ (allowing repetition) are equally likely. What is the probability that such a string has no a ?

b. [5] What is the probability that such a string has no b given that it has no a ?

6. [10] Given non-negative integers n_1, n_2, n_3 , and n_4 , how many distinct strings of length $n_1 + n_2 + n_3 + n_4$ are there that have exactly n_1 1's, n_2 2's, n_3 3's, and n_4 4's?

7. [10] How many triples $\langle n_1, n_2, n_3 \rangle$ of non-negative integers n_1, n_2 , and n_3 exist that satisfy $n_1 + n_2 + n_3 = 10$. (Hint: Think about balls and bins.)

8. [10] For $n \geq 1$, assume all strings of length n from the set $\{a, b, c\}$ (allowing repetition) are equally likely. What is the expected number of c 's.

1. [10] For $n \geq 3$, consider bit strings of length n . How many such strings begin with the substring 111, end with the substring 111, or both? (Do not ignore $n = 3, 4$, and 5.)

2. a. [10] Present a combinatorial argument that for all positive values of n :

$$3^n = \sum_{i=0}^n \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j}$$

b. [10] Present a combinatorial argument that for all m and n satisfying $2 \leq m$, $2 \leq n$, and $m \leq n + 1$:

$$\binom{n+2}{m} = \binom{n+1}{m} + \binom{n}{m-1} + \binom{n}{m-2}$$

(Hint: Consider $A = B \cup \{c\} \cup \{d\}$, where $c \neq d$, $c \notin B, d \notin B$, and $\#B = n$.)

3. a. [10] For $n \geq 1$, how many decimal numbers between 1 and $10^n - 1$ contain no 5's or 7's?

b. [10] For $n \geq 1$, how many decimal numbers between 1 and $10^n - 1$ contain at most one 5 and one 7?

4. [10] For $n \geq 1$, how many ways can you find ordered triples (i, j, k) so that i, j , and k are non-negative and their sum is n ? (Hint: Consider balls and bins.)

5. [10] For $n \geq 1$, assume all strings of n digits from $\{0, 1, \dots, 9\}$ are equally likely. What is the expected number of 9's in such a string?

6. [10] For $n \geq 3$, what is the probability that a string of length n of a 's, b 's, c 's, and d 's has three or more a 's (assuming all such strings are equally likely)?

7. [10] Consider two cards drawn from a 52 card deck and assume all such draws are equally likely. Is the event that a heart is drawn as the first card independent of the event that a spade is drawn of the second card?

1. [10] For $n \geq 4$, consider a set A of cardinality n . How many subsets of A are of cardinality less than or equal to 3?

2. a. [10] Present a combinatorial argument that for all n and k satisfying $1 \leq n$ and $k \leq n$:

$$n! = \binom{n}{k} \cdot k! \cdot (n - k)!$$

b. [10] Present a combinatorial argument that for all positive values of n :

$$2^n = 1 + \sum_{k=0}^{n-1} 2^k$$

(Hint: Consider Let k be the position of the first 1 in a bit string.)

3. a. [10] For $n \geq 1$, consider strings of length n from the set of characters $\{a, b, c, d, e, f\}$ allowing repetition. How many such strings at most one a and one b ?

b. [10] For $n \geq 1$, how many decimal numbers between 1 and $10^n - 1$ contain at most one 5 and one 7?

4. [10] For $m \geq n \geq 1$, in how many ways can you place m balls into n boxes so that every box has at least one ball?
5. [10] For $n \geq 1$, assume all strings of n digits from $\{0,1,\dots,9\}$ are equally likely. What is the expected number of 9's in such a string?
6. [10] For $n \geq 3$, what is the probability that a string of length n of a 's b 's, c 's, and d 's has three or more a 's (assuming all such strings are equally likely)?
7. [10] Consider two cards drawn from a 52 card deck and assume all such draws are equally likely. Is the event that a heart is drawn as the first card independent of the event that a spade is drawn of the second card?

1. [5] Consider integers in the set $\{1, 2, 3, \dots, 1000\}$. How many are divisible by either 4 or 10?

2. a. [10] Present a combinatorial argument that for all $n \geq 1$:

$$\sum_{k=0}^n \binom{n}{k} 2^k = 3^n$$

b. [10] Present a combinatorial argument that for all nonnegative integers p , s , and n satisfying $p + s \leq n$

$$\binom{n}{p} \binom{n-p}{s} = \binom{n}{p+s} \binom{p+s}{p}$$

(Hint: Consider choosing two subsets.)

3. [10] For $n \geq 1$, Let $\mathcal{A} = \{1, 2, \dots, 2n\}$. How many subsets of \mathcal{A} contain exactly k_1 even numbers and k_2 odd numbers?

4. [10] For $n \geq 1$, how many ordered triples (n_1, n_2, n_3) of non-negative numbers satisfy $n_1 + n_2 + n_3 = n$? (Hint: think about putting balls into bins.)

5. [10] For $n \geq 1$, assume all strings of n characters from $\{a, b, c, d\}$ are equally likely. What is the expected number of a 's in such a string?

6. [10] Given a finite event space E (in which all events are equally likely) and subsets A and B of E , show that

$$\Pr(A \cup B) \geq \Pr(A) + \Pr(B) - 1.$$

7. [10] Consider a 52 card deck of cards from which the ace of spades is removed resulting in a 51 card deck. Further, consider two distinct cards drawn from the 51 card deck and assume all such unordered draws are equally likely. Lastly consider the probability of the

event that both cards are hearts. Is it more likely that both cards are hearts if it is given that both cards are face cards (i.e., Kings, Queens, or Jacks)?

8. [10] Let \mathcal{A} be a set of cardinality p . Consider ordered strings of length m using the elements of \mathcal{A} . How many such strings have the m^{th} component a repetition of one of the preceding $m-1$? (Hint: Think about the complement and think about selecting the m^{th} component first.)

1. [5] Suppose all rolls of a six-side die are equally likely. What is the probability the roll is a six given that it is not one?

2. a. [10] Present a combinatorial argument that for all $n \geq 1$:

$$\sum_{k=1}^n \binom{n}{k} = 2^n - 1$$

(**Note:** The summation begins with $k = 1$.)

b. [10] Present a combinatorial argument that for all integers k and n satisfying $3 \leq k \leq n$

$$\binom{n}{k} = \binom{n-3}{k} + 3\binom{n-3}{k-1} + 3\binom{n-3}{k-2} + \binom{n-3}{k-3}$$

(**Hint:** Consider three special elements.)

3. [10] How many distinct permutations are there of the letters in “mississippi”?

4. [10] A bin has 100 blue balls, 100 red balls, and 100 green. How many different collections can I obtain using 100 of these balls? (Balls of the same color are indistinguishable from one another but are distinguishable from balls of another color. A “collection” has no order to it.)

5. [10] Assume all strings of length five using characters from $\{a, b, c, d\}$ are equally likely. What is the probability that there is a substring $\langle abc \rangle$ in the string?

6. [10] Suppose a number k from $\{1, 2, \dots, 100\}$ is to be drawn and that all numbers are equally likely. Let A be the event k is a power of two. Let B be the event k is an integer multiple of four. Prove either that the events A and B are statistically independent or that they are statistically dependent.

7. [10] For $n \geq 1$, consider strings of length n containing 0's and 1's but ending in a 1. Assume all such strings are equally likely. What is the expected number of 1's in such a string?

8. [10] For $n \geq 1$, how many strings of length n employing the characters $\{a, b, c\}$ have at least one a ?

1. [5] For $n \geq 3$, how many subsets of size 3 from $\{a_1, a_2, \dots, a_n\}$ are there that either contain a_1 or a_2 (or both)?
2. [10] Given a set A of m characters, for $n \geq 2$, consider strings of length n using any of the characters of A . How many such strings begin and end with the same character?
3. [10] Present a combinatorial argument that for all positive integers m, n , and r , satisfying $r \leq \min\{m, n\}$:

$$\binom{m+n}{r} = \sum_{k=0}^n \binom{m}{k} \binom{n}{r-k}.$$

(Hint: Consider selecting from two sets.)

- b. [10] Present a combinatorial argument that for all positive integers n :

$$3^n = \sum_{i=0}^n \left(\sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j} \right)$$

(Note: Be very specific about the roles of i and j .)

4. [10] How many distinct permutations are there of the digits in 1121231234?
5. [10] Given $n \geq r \geq 1$, in how many ways can n identical balls be placed into r distinct bins such that each bin contains at least one ball? (Hint: Consider strings with balls and special dividers.)
6. [10] For $n \geq 1$, consider strings of length $2n$ 0's and 1's. Assuming all such strings are equally likely, what is the probability that such a string has an equal number of 0's and 1's?
7. a. [10] For $n \geq 5$, consider strings of length n using elements of $\{a, b, c, d\}$. Assume all such strings are equally likely. What is the probability that a string has exactly two a 's?
 - b. [5] What is the probability that such a string has exactly three b 's given that it has exactly two a 's?