Examination 1 Solutions

1. [10] How many arrangements of the numbers 31415926535 begin with 3 or end with 5 (or both)?

In the array of 11 numbers there 2 ones, 1 two, 2 threes, 1 four, 3 fives, 1 six, and 1 nine. Of these, $\binom{10}{2\ 1\ 1\ 1\ 3\ 1\ 1}$ begin with a three, $\binom{10}{2\ 1\ 2\ 1\ 2\ 1\ 1}$ end with a five, and $\binom{9}{2\ 1\ 1\ 1\ 2\ 1\ 1}$ begin with a three and end with a five. By the principle of inclusion and exclusion $\binom{10}{2\ 1\ 1\ 1\ 3\ 1\ 1}+\binom{10}{2\ 1\ 2\ 1\ 2\ 1\ 1}-\binom{9}{2\ 1\ 1\ 1\ 2\ 1\ 1}$ begin with 3 or end with 5.

2. [10] A class of 40 students is to illustrate all functions from $\{a,b,c\}$ into $\{1,2,...,8\}$. Prove that at least one student must illustrate at least nine one-to-one functions.

There are $8\cdot 7\cdot 6=336$ one-to-one functions from $\{a,b,c\}$ into $\{1,2,...,8\}$. Since $\left\lceil \frac{336}{40} \right\rceil = \lceil 8.6 \rceil = 9$, by the pigeonhole principle, at least one student must illustrate at least nine one-to-one functions.

3. a. [10] Using a combinatorial argument, prove that for $n \geq 1$ and $k \geq 1$:

$$n^k - n^{k-1} = (n-1)n^{k-1}$$

Consider arrays of length k selected from a set $\{a_1, a_2, ..., a_n\}$ in which the first element cannot be a_1. For the left side, there are n^k total arrays without the restriction and n^{k-1} arrays that have a_1.as the first element. Thus, there are $n^k - n^{k-1}$ arrays in which the first element is not a_1. Alternatively, there are $n-1$ non-a_1 options for the first elements and n options for the remaining $k-1$ elements, giving $(n-1)n^{k-1}$. We may conclude that $n^k - n^{k-1} = (n-1)n^{k-1}$.

b. [10] Using a combinatorial argument, prove that for $m \geq n \geq p \geq 0$:

$$\binom{m}{n}\binom{n}{p} = \binom{m}{p}\binom{m-p}{n-p}$$

Consider selecting two distinct subsets, A and B, of cardinalities $n-p$ and p, respectively, from a set C of cardinality m. For the left side, there are $\binom{m}{n}$ ways to select $A \cup B$ from C, then $\binom{n}{p}$ ways to select B from $A \cup B$. The remaining elements of $A \cup B$ become A. Thus, there are $\binom{m}{n}\binom{n}{p}$ such decompositions. Alternatively, we may select the elements of B first in $\binom{m}{p}$ ways and then select the $n-p$ elements of A from the remaining $m-p$ elements of C □ B. This can be done in $\binom{m-p}{n-p}$ ways, so there are there are $\binom{m}{p}\binom{m-p}{n-p}$ such selections and this must equal $\binom{m}{n}\binom{n}{p}$.

4. [10] For $n \geq 2$, how many arrays of length $2n-1$ using one a, one b, and $2n-3$ $c's$ have the a in one of first n positions and the b in one of last n positions? (Obviously only one could occupy position n.)

Assuming position n is occupied by a c, there are $n-1$ positions for the a and $n-1$ positions for the b, so $(n-1)^2$ arrays. If either a or b occupies position n, there are two choices for which of those is in the position and then $n-1$ choices for the position of the one not in position n, so $2(n-1)$ arrays. The total is $(n-1)^2 + 2(n-1) = (n+1)(n-1)$ arrays.

5. [10] How many numbers between 0 and 999,999 have (decimal) digits adding to 8?

Consider six bins labeled 1 through 6. If balls are placed into the bins and no more than nine balls are in any one bin, we could interpret the result as a number with the k^{th} digit equaling the number of balls in the k^{th} bin. Thus, no balls in any bin is associated with 0, ..., nine balls in each of the six bins is 999,999. The sum of the digits is the total number of balls. If 8 balls are positioned in the six bins the digits of the associated numbers ad to 8. There are $\binom{8+6-1}{8} = \binom{13}{8}$ such numbers.

6. a. [5] Two distinct, unordered numbers are selected from $\{1,\ldots,30\}$ and all such selections are equally likely. What is the probability that (at least) one of the numbers is odd?

There are $\binom{30}{2}$ equally likely selections but since there are 15 even numbers in $\{1,\ldots,30\}$, there are $\binom{15}{2}$ selections where both numbers are even so there are $\binom{30}{2}-\binom{15}{2}$ selections where at least one number is odd. The probability is then

$$\frac{\binom{30}{2}-\binom{15}{2}}{\binom{30}{2}}.$$

b. [5] What is the probability that both numbers are prime given that (at least) one of them is odd?

There are 9 odd primes in $\{1,\ldots,30\}$. Any selection of two distinct primes from $\{1,\ldots,30\}$ must have at least one odd number (since there is only one even prime). So there are $\binom{10}{2}$ selections where both numbers are prime and at least one is odd. The probability that both numbers are prime given that (at least) one of them is odd is $\dfrac{\binom{10}{2}}{\binom{30}{2}-\binom{15}{2}}$.