Examination 1 Solutions

CS 336

1. For $n \geq m \geq 1$, let $A = \{1, 2, ..., 2m\}$, $B = \{1, 2, ..., 2n\}$, and consider functions mapping from A into B.

a. [5] How many of these functions map even numbers to even numbers and odd number to odd numbers (i.e., how many functions have **both** properties)?

For each of the m *even numbers in* A *there are* n *even numbers in* B *to which they can be mapped. Thus there are* n^m *such choices. Similarly, for each of the* m *odd numbers in* A *there are* n *oddn numbers in* B *to which they can be mapped. Thus again there are* n^m *such choices. There are* $n^m n^m = n^{2m}$ *such functions.*

b. [5] Of these functions that map even numbers to even numbers and odd number to odd numbers, how many are one-to-one?

There are n *even numbers in* B *to which 2 can be mapped, then* $n-1$ *even numbers in* B *to which 4 can be mapped,…, and* $n-(k-1)$ *even numbers in* B *to which 2k can be mapped, in general for* $k = 1, 2, ..., n$. *Thus there are* $n(n-1)\cdots(n-m+1)$ *such choices. Similarly, for the odd numbers in A there are* $n(n-1)\cdots(n-m+1)$ *such choices. There are* $n^2(n-1)^2\cdots(n-m+1)^2 = (\frac{n!}{(n-m)!})^2$ *such functions.*

2.a [10] Present a combinatorial argument that for all positive integers n:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

Let set $C = A \cup B$, where A and B are disjoint and have n elements each. We consider the number of permutations of subsets of C of size 2. Since there are $2n$ elements in C there are $\binom{2n}{2}$ such subsets. Alternatively, the two elements could come from a single set, A or B, or one element could come from each. If the two elements come from a single set, there are two choices for the set and then $\binom{n}{2}$ choices for the subset. That's a total of $2\binom{n}{2}$ chosen in that manner. If one element comes from each, there n^2 possibilities. The overall total is $2\binom{n}{2} + n^2$ subsets of size 2 from the elements of C and this must equal $\binom{2n}{2}$.

b. [10] Present a combinatorial argument that for all positive integers *n*

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

(Hint: Consider distinct sets A and B each of cardinalities *n*.)

Let A and B be disjoint sets of cardinality n each. Consider how many subsets of $A \cup B$ there are of cardinality n. Since we are selecting without repetition and ignoring order from a set of cardinality $2n$, there are $\binom{2n}{n}$ such subsets. Alternatively, separate this into cases based upon the number of elements of A in the subset. This value, denoted by k, can vary from zero to n. For a fixed value of k, there are $\binom{n}{k}$ ways to select the elements of A, and then $\binom{n}{k}$ ways to select the elements of B that will **not** *be in the subset - leaving the $n-k$ that* **wil***l be in the subset. Thus there are $\binom{n}{k}^2$ way to choose the subset with k elements coming from A. The total is then $\sum_{k=0}^{n} \binom{n}{k}^2$ and this must equal $\binom{2n}{n}$.*

3. [10] For $n \geq m \geq 1$, in how many ways can n **identical** coins be distributed among m **non-identical** people such that every person has at least one coin?

Consider giving each person one coin and then distributing the $n-m$ *remaining coins. This is equivalent to distributing* $n-m$ *identical balls into* m *bins. There are* $\binom{(n-m)+(m-1)}{m-1} = \binom{n-1}{m-1} = \binom{n-1}{n-m}$ *ways to arrange the balls so this is the number of* ways n **identical** coins be distributed among m **non-identical** people such that every person has at least one coin.

4. [10] For $n \geq n_1, n_2, n_3, n_4 \geq 0,$ you are given n non-identical books and five non-identical boxes. How many ways are there to distribute books into the boxes so that box 1 has exactly n_1 books, box 2 has exactly n_2 books, …, and box 5 has the remaining books (if any)?

The are $\binom{n}{n_1}$ *ways of selecting the books for box 1. Given those, there are* $\binom{n-n_1}{n_2}$ *ways of selecting the books for box 2, then* $\binom{n-n_1-n_2}{n_3}$ *ways of selecting the books for box 3, and* $\binom{n-n_1-n_2-n_3}{n_4}$ *ways of selecting the books for box 4. All remaining books (if any) are forced to go into box 5. Thus, there are* $\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\binom{n-n_1-n_2-n_3}{n_4} = \binom{n}{n_1 \quad n_2 \quad n_3 \quad n_4}$ *ways of distributing the books.*

5. [10] For $n \geq 3$, how many strings of length n consisting of *a*s, *b*s, and *c*s are there that have exactly one *a* and at least two bs?

We will consider the number of strings with exactly one a and then subtract from this the number of strings with exactly one a and no bs, and the number of strings with exactly one a and one b. There are n *positions for the single a and then each of the other* $(n-1)$ *positions has two options. Thus, there are* $n2^{n-1}$ *strings with exactly one a. Of there, if there is exactly one a and no bs there just* n *possibilities. Finally, if , if there is exactly one a and one b there just* n *positions for the a, then* $n-1$ *positions for the b (the rest must be cs) Thus there are* $n(n-1)$ *strings with exactly one a and one b. By differencing, the number of strings with exactly one a and at least two bs is* $n2^{n-1} - n - n(n-1) = n2^{n-1} - n^2$.

6. Consider strings of length $n \geq 5$ containing exactly k 1s and $n-k$ 0s, where $k \geq 5$. Consider that all such strings are equally likely.

a. [5] What is the probability that such a string begins with five 1s?

There are $\binom{n}{k}$ *such equally likely strings. Of these,* $\binom{n-5}{k-5}$ *begin with five 1s so the probability of a string beginning with five ones is* $\binom{n-5}{k-5} / \binom{n}{k}$.

b. [5] What is the probability that such a string begins with five 1s given that it begins with three 1s?

There are $\binom{n-3}{k-3}$ *equally likely strings that begin with three 1s.. Of these,* $\binom{n-5}{k-5}$ *begin with five 1s so the probability of a string beginning with five ones is* $\binom{n-5}{k-5} / \binom{n-3}{k-3}$.