Examination 1 Solutions

CS 336

1. For $n \geq m \geq 1$, let $A = \{1, 2, ..., 2m\}$, $B = \{1, 2, ..., 2n\}$, and consider functions mapping from A into B.

a. [5] How many of these functions map even numbers to even numbers and odd number to odd numbers (i.e., how many functions have **both** properties)?

b. [5] Of these functions that map even numbers to even numbers and odd number to odd numbers, how many are one-to-one?

2.a [10] Present a combinatorial argument that for all positive integers n:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

b. [10] Present a combinatorial argument that for all positive integers n

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

(Hint: Consider distinct sets A and B each of cardinalities n.)

3. [10] For $n \geq m \geq 1$, in how many ways can n **identical** coins be distributed among m **non-identical** people such that every person has at least one coin?

4. [10] For $n \geq n_1, n_2, n_3, n_4 \geq 0$, you are given n non-identical books and five non-identical boxes. How many ways are there to distribute books into the boxes so that box 1 has exactly n_1 books, box 2 has exactly n_2 books, ..., and box 5 has the remaining books (if any)?

5. [10] For $n \geq 3$, how many strings of length n consisting of *a*s, *b*s, and *c*s are there that have exactly one *a* and at least two bs?

6. Consider strings of length $n \geq 5$ containing exactly k 1s and $n-k$ 0s, where $k \geq 5$. Consider that all such strings are equally likely.

a. [5] What is the probability that such a string begins with five 1s?

b. [5] What is the probability that such a string begins with five 1s given that it begins with three 1s?