Examination 1 Solutions

CS 336

1. For \( n \geq m \geq 1 \), let \( A = \{1, 2, \ldots, 2m\} \), \( B = \{1, 2, \ldots, 2n\} \), and consider functions mapping from \( A \) into \( B \).

a. [5] How many of these functions map even numbers to even numbers and odd number to odd numbers (i.e., how many functions have both properties)?

b. [5] Of these functions that map even numbers to even numbers and odd number to odd numbers, how many are one-to-one?

2. [10] Present a combinatorial argument that for all positive integers \( n \):

\[
\binom{2n}{2} = 2 \binom{n}{2} + n^2.
\]

b. [10] Present a combinatorial argument that for all positive integers \( n \)

\[
\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.
\]

(Hint: Consider distinct sets \( A \) and \( B \) each of cardinalities \( n \).)

3. [10] For \( n \geq m \geq 1 \), in how many ways can \( n \) identical coins be distributed among \( m \) non-identical people such that every person has at least one coin?

4. [10] For \( n \geq n_1, n_2, n_3, n_4 \geq 0 \), you are given \( n \) non-identical books and five non-identical boxes. How many ways are there to distribute books into the boxes so that box 1 has exactly \( n_1 \) books, box 2 has exactly \( n_2 \) books, \( \ldots \), and box 5 has the remaining books (if any)?

5. [10] For \( n \geq 3 \), how many strings of length \( n \) consisting of \( a \)'s, \( b \)'s, and \( c \)'s are there that have exactly one \( a \) and at least two \( b \)'s?

6. Consider strings of length \( n \geq 5 \) containing exactly \( k \) 1's and \( n - k \) 0's, where \( k \geq 5 \). Consider that all such strings are equally likely.

a. [5] What is the probability that such a string begins with five 1's?

b. [5] What is the probability that such a string begins with five 1's given that it begins with three 1's?