Examination 1 Solutions

CS 336

1. [5] How many strings of *0*s and *1*s of length 8 have an even number of *0*s or an even number of *1*s or both?

Since the length is eight, having an even number of 0s is equivalent to having an even number of 1s. The number of 0s is either 0, 2, 4, 6, or 8. Thus, there are $\binom{8}{0}+\binom{8}{2}+\binom{8}{4}+\binom{8}{6}+\binom{8}{8}$ *ways to choose the places for them. The 1s fill the remaining places.*

2 a. [5] For $n \geq 2$, how many permutations of $a_1, a_2, ..., a_n$ have a_1 in positions 1, 2, or 3?

There are three places for a_1. *This leaves* $n-1$ *places for* $a_2,...,a_n$ *and they can be filled in* $(n-1)!$ *way, so there are* $3(n-1)!$ *permutations of* $a_1, a_2, ..., a_n$ *have* a_1 *in positions 1, 2, or 3.*

[5] b. For $n \geq 2$, how many permutations of $a_1, a_2, ..., a_n$ have a_1 in positions 1, 2, or 3 and a_2 in positions 2, 3, or 4?

We must examine cases. If a_1 *is in position 1, then there are three options for* a_2. *Alternatively, if* a_1 *is in positions 2 or 3, then there are only two options (each) for* a_2. *The total is* $3+2\cdot 2=7$. *Having positioned* a_1 *and* a_2, *there remain* $n-2$ *places for* $a_3,...,a_n$ *and they can be filled in* $(n-2)!$ *way. Thus, there are* $7(n-2)!$ *permutations of* $a_1, a_2, ..., a_n$ *having* a_1 *in positions 1, 2, or 3 and* a_2 *in positions 2, 3, or 4.*

3.a [10] Present a combinatorial argument that for all positive integers *n*

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

(Hint: Consider distinct sets A and B each of cardinalities *n*.)

Let A and B be disjoint sets of cardinality n each. Consider how many subsets of $A \cup B$ *there are of cardinality n. Since we are selecting without repetition and ignoring order from a set of cardinality* $2n$*, there are* $\binom{2n}{n}$ *such subsets. Alternatively, separate this into cases based upon the number of elements of* A *in the subset. This value, denoted by k, can vary from zero to n. For a fixed value of k, there are* $\binom{n}{k}$ *ways to select the elements of A, and then* $\binom{n}{k}$ *ways to select the elements of B that will* **not** *be in the subset - leaving the* $n-k$ *that* **wil***l be in the subset. Thus there are* $\binom{n}{k}^2$ *way to choose the subset with k elements coming from* A*. The total is then* $\sum_{k=0}^{n} \binom{n}{k}^2$ *and this must equal* $\binom{2n}{n}$.

b [10] Present a combinatorial argument that for all positive integers $1 \le k < m$:

$$m! = \binom{m}{k} k!(m-k)!.$$

Let set $C = A \cup B$*, where A and B are disjoint and have k and (m-k) elements, respectively. We consider the number of permutations of* C*. Since there are* m *elements in C there are* $m!$ *permutations. Alternatively, first choose the* k *positions to be occupied by the k elements of A. There are* $\binom{m}{k}$ *ways to choose these. Then, permute these elements of A. There are* $k!$ *ways to do this. Finally, the remaining m-k position must hold the elements of B and there are* $(m-k)!$ *was to permute them. This leaves* $\binom{m}{k} k!(m-k)!$ *permutations of the elements of C and this must equal* $m! = \binom{m}{k} k!(m-k)!$

4. [10] Consider 5-tuples of the form $<r_1, r_2, r_3, r_4, r_5>$, where the $r_i \geq 0$. How many such 5-tuples are there satisfying

$$r_1 + r_2 + r_3 + r_4 + r_5 \leq 20?$$

Consider arranging 20 identical balls in six bins labeled r_1, r_2, r_3, r_4, r_5 *and "excess". The number of balls in each bin is non-negative and the total number is 20 so the sum of those in the first five bins is at most 20. There are* $\binom{20+6-1}{20}$ *ways to arrange the balls so this is the number of 5-tuples are there satisfying* $r_1 + r_2 + r_3 + r_4 + r_5 \leq 20$.

5. [10] For $n \geq 5$, consider strings of length n using upper case roman letters (i.e., {*A, B, C,…,X,Y,Z*}). Assuming all such strings are equally likely what is the probability that the string occurs in non-decreasing order (i.e. all *A*s precede all *B*s, all *B*s precede all *C*s ,… etc.)?

The are 26^n *such equally likely strings. The number of these strings in which the characters occur in non-decreasing order is the same as the number of ways of placing* n *balls in 26 bins (since the strings is totally determined by knowing the number of A's. B's, etc.) There are* $\binom{26+n-1}{n}$ *such placements so the probability of a non-decreasing string is* $\binom{26+n-1}{n} / 26^n$.

6. [10] Consider strings of length $n \geq 2$ containing exactly k 1's and $n-k$ 0's and having no adjacent 1's (i.e., there is at least one 0 between any 1's). Assuming $k \geq 1$ and $n \geq 2k$, how many such strings are there? (Hint: Consider cases based upon the contents of the last position.)

The string must terminate in a 1 or a 0. If it terminates in a 1, then all $k-1$ *previous 1's must be immediately followed by 0's (i.e 10 pairs). This leaves* $n-1-2(k-1) = n-2k+1$ *0's not immediately following 1's. With* $k-1$ *10 pairs and* $n-2k+1$ *0's, there are* $n-k$ *positions to be filled. This can be done in* $\binom{n-k}{k-1}$ *ways. If the string terminates in a 0, then all* k *previous 1's must be immediately followed by 0's (i.e 10 pairs). This leaves* $n-2k$ *0's not immediately following 1's. With* k *10 pairs and* $n-2k$ *0's, there are* $n-k$ *positions to be filled. This can be done in* $\binom{n-k}{k}$ *ways. The total is then* $\binom{n-k}{k-1} + \binom{n-k}{k}$.