Examination 1 Solutions

CS 336

1. [5] Given $n \geq 9$, how many strings of length n containing *0*s, *1*s, *2*s, and *3*s, have exactly three *0*s, exactly four *1*s, and either one or two *2*s (and the rest *3*s)?

There are $\binom{9}{3\ \ 4\ \ 1}$ *strings with exactly three 0s, exactly four 1s, one 2s, and the rest 3s and* $\binom{9}{3\ \ 4\ \ 2}$ *strings with exactly three 0s, four 1s, two 2s, and the rest 3s. Thus there are* $\binom{9}{3\ \ 4\ \ 1}+\binom{9}{3\ \ 4\ \ 2}$ *containing 0s, 1s, 2s, and 3s, with exactly three 0s, exactly four 1s, and either one or two 2s (and the rest 3s).*

2. For $n \geq 1$, let A and B be disjoint sets, each of cardinality n, and $C = A \cup B$. Consider functions $f : C \to C$.

[5] a. How many such functions are there that map A to B and B to A (i.e., for all $x \in C$, if $x \in A$ then $f(x) \in B$ and if $x \in B$ then $f(x) \in A$)?

For each element $a \in A$ *there are* n *options for* $f(a)$ *and for each element* $b \in B$ *there are* n *options for* $f(b)$, *so* $2n$ *times there are* n *options. The number of functions that map* A *to* B *and* A *to* B *is* n^{2n}.

[5] b. How many such **one-to-one** functions are there that map A to B and B to A?

Without loss of generality, let $A = \{a_1, a_2, ..., a_n\}$. *For* a_1 *there are* n *options for* $f(a_1)$. *For* a_2 *there are* $n-1$ *options for* $f(a_2)$ *since* $f(a_2) \neq f(a_1)$. *In general, for* a_k *there are* $n-(k-1)$ *options for* $f(a_k)$ *since* $f(a_k) \notin \{f(a_1), ..., f(a_{k-1})\}$. *We find that there are then* $n!$ *options for the values of* $f(a_1), ..., f(a_{n-1})$ *and* $f(a_n)$. *Similarly and independently, the are* $n!$ *options for the values of* $f(b_1), ..., f(b_{n-1})$ *and* $f(b_n)$ *so there are* $(n!)^2$ *one-to-one functions that map* A *to* B *and* B *to* A *?*

3.a [10] Present a combinatorial argument that for all positive integers $n, p,$ and q:

$$\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k} = (p+q)^n.$$

Let A and B be disjoint sets of cardinality p and q, respectively, and $C = A\cup B$. Consider how many strings of length n there are composed of elements of C. Let k denote the number of positions in the string occupied by elements of A. Clearly k may vary from 0 to n. For a fixed value of k, there are $\binom{n}{k}$ ways to select the positions to contain the elements of A and the p options for each of the k positions. The remaining $n-k$ positions must hold elements of B. There are q options for each of the $n-k$ positions. Thus, for a fixed value of k, there are $\binom{n}{k}p^k q^{n-k}$ such strings and $\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k}$ overall. Alternatively, for each of the n positions there are $p+q$ options for a total of $(p+q)^n$ strings, and this must equal $\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k}$.

b [10] Present a combinatorial argument that for all integers p and n so that $1 \le p \le n$:

$$\binom{n+3}{p} = \binom{n}{p} + \binom{3}{1}\binom{n}{p-1} + \binom{3}{2}\binom{n}{p-2} + \binom{n}{p-3}.$$

Let set $C = A\cup B$, where A and B are disjoint and have n and 3 elements, respectively. Thus the cardinality of C is $n+3$. We consider subsets of C of size p. Since there are $n+3$ elements in C there are $\binom{n+3}{p}$ such subsets. Alternatively, consider cases based upon the number of elements of set B in the subset. This number must be either zero, one, two, or three. If zero, all p elements must came from A so there are $\binom{n}{p}$ subsets. If one, there are $\binom{3}{1}$ ways to chose that element and $\binom{n}{p-1}$ ways to chose the remaining $p-1$ elements of set A. Thus, there are $\binom{3}{1}\binom{n}{p-1}$ subsets. If two, there are $\binom{3}{2}$ ways to chose these elements and $\binom{n}{p-2}$ ways to chose the remaining $p-2$ elements of set A. Thus, there are $\binom{3}{2}\binom{n}{p-2}$ subsets. Finally, if all elements of elements set B are used then $p-3$ must come from A so there are $\binom{n}{p-3}$ subsets. The total is $\binom{n}{p} + \binom{3}{1}\binom{n}{p-1} + \binom{3}{2}\binom{n}{p-2} + \binom{n}{p-3}$ subsets and this must equal $\binom{n+3}{p}$.

4. [10] Given positive integers p and n, in how many ways can p identical tokens be distributed to n different people so that no person has all of the tokens?

Consider arranging p identical balls(tokens) in n bins (people). There are $\binom{n+p-1}{p}$ *ways to arrange the balls. However for each bin there is one distribution in which that bin has all of the balls and this is disallowed. Thus, there are* $\binom{n+p-1}{p}-n$ *ways in which p identical tokens be distributed to n different people so that no person has all of the tokens.*

5. Consider **six** card hands drawn for a 52 card deck and assume all such are equally likely.

a.[5] What is the probability that the hand has exactly five clubs?

There are $\binom{52}{6}$ *such equally likely hands. The number of these hands having exactly five clubs is* $39\binom{13}{5}$ *since there are* $\binom{13}{5}$ *ways to select the five clubs from the 13and 39 non-clubs left in the deck. The probability that the hand has exactly five clubs is then* $39\binom{13}{5}/\binom{52}{6}$.

a.[10] What is the probability that the hand has exactly five clubs **given that it has at least one spade**?

The number of hands having no spade is $\binom{39}{6}$, *so there* $\binom{52}{6}-\binom{39}{6}$ *equally likely hands with at least one spade. The number of these hands having exactly five clubs and at least one spade (therefore, exactly one spade) is* $13\binom{13}{5}$ *since there are* $\binom{13}{5}$ *ways to select the five clubs from the 13 and 13 spades in the deck. The probability that the hand has exactly five clubs given that it has at least one spade is then* $13\binom{13}{5}/(\binom{52}{6}-\binom{39}{6})$.