Examination 2 Solutions

CS 336

1. [20] Using only Definition 2', prove that the set S of odd length strings of *a*s is infinite**.** (For uniformity, please use the $<...>$ notation for strings (so $<aaa>$ is a string of three *a*s) and $||$ for concatenation of strings (so $<aa>||<aaaa>=<aaaaaa>$).)

For $s \in S$, define $f: S \to S$ by $f(s) = <aa>||s$ (i.e. the string consisting of two *a*s concatenated with s). If $s \in S$ then s has an odd number of *a*s so $f(s) = <aa>||s$ will as well. For $s, t \in S$, if $s \neq t$, then s and t must have different lenths. Thus $f(s) = <aa>||s \neq <aa>||t = f(t)$ since $<aa>||s$ and $<aa>||t$ must also have different lengths, so f is one-to-one. Since for all $s \in S$, $f(s)$ must have at least two *a*s, there is no $s \in S$ such that $f(s) = <a>.$. We have then that f maps S into $S \sim \{<a>\}$, which is a proper subset of S, and by Definition 2', S is infinite.

2. [20] Consider the set P of strings of {a,b,c,d} of odd length. Prove P is countably infinite.

For $n \in \mathbb{N}$, let $P_n = \{strings\ of\ \{a,b,c,d\}\ of\ length\ 2n+1\}$. So $|P_n| = 4^n$ and each P_n is finite. Since $P = \bigcup_{n \in \mathbb{N}} P_n$, by Theorem 10, P is countable. By Theorem 1, $\mathbb{N}$ is infinite and we define $f: \mathbb{N} \to P$ by $f(n) = <a...a>$ of length $2n+1$. For $n, m \in \mathbb{N}, n \neq m \Rightarrow 2n+1 \neq 2m+1 \Rightarrow f(n) \neq f(m)$ since the first has length $2m+1$ and the second has length $2m+1$, so f is one to one and thus by Theorem 4, P is infinite. We conclude that P is countable infinite.

3. [20] Let A be finite (and non-empty), B be countably infinite, C be uncountably infinite and $D = A \times B \times C$. Is D finite, countably infinite, or uncountably infinite? Prove your claim.

The set D is uncountably infinite. Let $a \in A$ and $b \in B$. Define $f: C \to D$ by $f(c) = (a,b,c)$. This function is one-to-one since if c and c' are elements of C and $c \neq c'$, then $f(c) = (a,b,c) \neq (a,b,c') = f(c')$. By Theorem 12, D is uncountably infinite.

4. [20] Using no other asymptotic dominance theory than definitions, prove that $1 + 2n + 3n^2 + 4n^3 = O(n^3)$.

Let $M = 10$ and $N = 1$. For $n \geq N = 1$, we have

$|1 + 2n + 3n^2 + 4n^3| = 1 + 2n + 3n^2 + 4n^3 \leq 1n^3 + 2n^3 + 3n^3 + 4n^3 = 10n^3 = M|n^3|$.

We conclude $1 + 2n + 3n^2 + 4n^3 = O(n^3)$.

5. [20] Prove that if $f_1 = o(g)$ and $f_2 = o(g)$ then $f_1 + f_2 = o(g)$. (Hint: $\varepsilon = \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$.)

If $f_1 = o(g)$ then for if ε is positive then so is $\frac{\varepsilon}{2}$, so there exists N_1 so that $n \geq N_1$ guarantees $|f_1(n)| \leq \frac{\varepsilon}{2}|g(n)|$. Similarly, there exists N_2 so that $n \geq N_2$ guarantees $|f_2(n)| \leq \frac{\varepsilon}{2}|g(n)|$. Combining these we have for

$n \geq \max\{N_1, N_2\}$ $|f_1(n) + f_2(n)| \leq |f_1(n)| + |f_2(n)| \leq \frac{\varepsilon}{2}|g(n)| + \frac{\varepsilon}{2}|g(n)| = \varepsilon|g(n)|$.

We conclude that $f_1 + f_2 = o(g)$.

6. [20] Prove that if $0 < a < b$, then $b^n \neq \mathrm{O}(a^n)$.

Given $M \geq 0$ and $N \geq 0$, let $\overline{M} = \max\{M, 1\}$ thus $\overline{M} \geq M$ and $\ln(\overline{M}) \geq 0$. Notice that $\ln(\frac{b}{a}) > 0$ and choose $n = \max\{N, \left\lceil \frac{\ln(\overline{M})}{\ln(\frac{b}{a})} \right\rceil\} + 1$. For this *n* we have $n > \frac{\ln(\overline{M})}{\ln(\frac{b}{a})}$, thus $n\ln(\frac{b}{a}) > \ln(\overline{M})$ and $(\frac{b}{a})^n > \overline{M} \geq M$. But then $\left|b^n\right| = b^n > M\,a^n = M\left|a^n\right|$ so $b^n \neq \mathrm{O}(a^n)$.