Examination 2 Solutions

CS 336

1. [20] Using only Definition 2', show that the set positive, integral multiples of three (i.e., $\{3, 6, 9, \ldots\} = \{3k \mid k \in \mathrm{N} \text{ and } k \geq 1\}$) is infinite**.**

Let $A = \{3k \mid k \in \mathrm{N} \text{ and } k \geq 1\}$ and consider $f : A \to A$ defined by $f(a) = a + 3$ for $a \in A$. This function maps A into A since $a \in A \Rightarrow a + 3 \in A$. Furthermore, this function is one-to-one since for $a, b \in A$, if $a \neq b$, then $f(a) = a + 3 \neq b + 3 = f(b)$. The function f maps A into a proper subset of itself since $3 \in A$, but if $f(a) = a + 3 = 3$, then $a = 0$ but $0 \notin A$. We conclude no element of A maps to 3. By Definition 2', A is infinite

2. [20] Consider the set $A = \{(p, q, r) \mid p, q, r \in \mathrm{N}\}$ (i.e., ordered triples of natural numbers). Is A finite, countably infinite, or uncountably infinite? Prove your claim.

The set A is countably infinite. Recognize that $A = \mathrm{N} \times \mathrm{N} \times \mathrm{N}$. By Corollary 10.3 $\mathrm{N} \times \mathrm{N}$ is countably infinite and using it once more $\mathrm{N} \times \mathrm{N} \times \mathrm{N} = (\mathrm{N} \times \mathrm{N}) \times \mathrm{N}$ is counyably infinite.

3. [20] Show that the set of points in the square $B = \{(x, y) \mid -1/2 \leq x \leq 1/2 \text{ and } 1 \leq y \leq 2\}$ is uncountably infinite.

Consider the mapping $f : [0,1] \to B$ defined by $f(x) = (x - 1/2, 1)$. If $0 \leq x \leq 1$, then $-1/2 \leq x - 1/2 \leq 1/2$ and $1 \leq 1 \leq 2$ and thus f does map [0,1] into B. The function is one-to-one since if $x_1, x_2 \in [0,1]$:

$$f(x_1) = (x_1 - 1/2, 1) \neq (x_2 - 1/2, 1) = f(x_2)$$

By Theorem 11, B is uncountably infinite since $[0,1]$ is.

4. [20] Using only Definition 1, prove that for any $a \geq 0$, $n^4 = \mathrm{O}(n^{4.01})$.

We use $M = 1$ and $N = 0$. For $n \geq 0$, $n^{-.01} \leq n^0 = 1$, and $\left|n^4\right| = \left|n^{-.01}\, n^{4.01}\right| = \left|n^{-.01}\right|\left|n^{4.01}\right| \leq 1 \cdot \left|n^{4.01}\right|$. Therefore, $n^4 = \mathrm{O}(n^{4.01})$.

5. **[20]** Given $f = \mathrm{O}(g)$ and $g = o(h)$, either proof that $f = o(h)$ or construct a simple counterexample to prove that f is not necessarily $o(h)$.

Since $f = \mathrm{O}(g)$, there exist M_1 and N_1 so that $n \geq N_1 \Rightarrow |f(n)| \leq M_1 |g(n)|$. Given $\varepsilon > 0$, let $\varepsilon' = \varepsilon / M_1$. Since $g = o(h)$, there exist N_2 such that $n \geq N_2 \Rightarrow |g(n)| \leq \varepsilon' |h(n)| = \varepsilon / M_1 |h(n)|$. Thus letting $N = \max\{N_1, N_2\}$, for $n \geq N$ we have $|f(n)| \leq M_1 |g(n)| \leq M_1 \varepsilon / M_1 |h(n)| = \varepsilon |h(n)|$ so $f = o(h)$.

6. **[20]** Prove that for $0 < b < a$, $a^n \neq O(b^n)$.

Given any non-negative constants M and N, choose $n = \max\{N, \ln M / \ln(a/b) + 1\}$, then $n \geq N$ and $n > \ln M / \ln(a/b)$. We then have $\ln(a/b)^n = n \ln(a/b) > \ln M$ so $a^n / b^n = (a/b)^n > \ln M$. We conclude that $|a^n| > \ln M |b^n|$.