Examination 2 Solutions

1. [20] Using only Definition 2', show that the set of odd integers (i.e., {…, -3, -1, 1, 3, …} is infinite**.**

Let $A = \{2k+1 \mid k \in \mathbb{Z}\}$ and consider $f : A \to A$ defined by $f(a) = \begin{cases} a & a < 0 \\ a+2 & a > 0 \end{cases}$ for $a \in A$. To show this function is one-to-one consider $a, b \in A$: if $a \neq b$, then either $a,b < 0, a < 0 < b$, or $a,b > 0$. If $a,b < 0$, $f(a) = a \neq b = f(b)$. If $a < 0 < b$, $f(a) = a < b < b+2 = f(b)$. Finally, if $a,b > 0$, $f(a) = a+2 \neq b+2 = f(b)$. Thus, f is one-to-one. Lastly, the function f maps A into a proper subset of itself since $1 \in A$, but if $f(a) = 1$, then either $a < 0$ in which case $f(a) = a < 0 < 1$, or $a > 0$ in which case $f(a) = a+2 > 2 > 1$. Either of these leads to a contradiction, so we conclude no element of A maps to 1.

2. [20] Suppose the set U is uncountably infinite, the set V is countable and W is the set difference $U \sim V$. Prove or disprove (with a simple counter example):

W is uncountably infinite.

Since $W = U \sim V$, we have $U = W \cup V$. If W were countable then U would be the union of two countable sets. By Corollary 10.1 ,U would also be countable contrary to hypothesis. We conclude that W is uncountably infinite.

3. [20] Prove that the set of complex numbers $C = \{x + iy \mid x, y \in \mathbb{R}\}$ is uncountably infinite.

Consider the mapping $f : [0,1] \to C$ defined by $f(x) = x + 0i$. The function is one-to-one since if $x \in [0,1]$, $f(x) = x + 0i \neq y + 0i = f(y)$. By Theorem 11, C is uncountably infinite since $[0,1]$ is.

4. [20] Using only Definition 1, prove that $1 + 2n + 3n^2 = \mathrm{O}(n^2)$.

We use $M = 6$ and $N = 1$. For $n \geq 1$, $1 \leq n \leq n^2$, and $\left|1 + 2n + 3n^2\right| = 1 + 2n + 3n^2 \leq 6n^2 = 6\left|n^2\right|$. Therefore, $1 + 2n + 3n^2 = \mathrm{O}(n^2)$.

5. [20] Given that a function $f : \mathbb{N} \to \mathbb{R}$ assumes only positive values (i.e., $f(n) > 0$ for all $n \in \mathbb{N}$) and that $f^2 = \mathrm{O}(f)$ prove that $f = \mathrm{O}(1)$.

Since $f^2 = \mathrm{O}(f)$ we have that for some M and N, $n \geq N \Rightarrow | f(n)^2 | \leq M \mid f(n) |$. We also have that $f(n) \neq 0$, so we may divide by $f(n)$, thus $n \geq N \Rightarrow | f(n) | \leq M \, |1|$ and we conclude that $f = \mathrm{O}(1)$.

6. **[20]** Prove that for $0 < a < 1$, $a^n \neq O(a^{2n})$.

Given any non-negative constants M and N, notice that $a^{-1} > 1$ and choose $n = \max\{N, \log(M)/\log(a^{-1}) + 1\}$, then $n \geq N$ and $n > \log(M)/\log(a^{-1}) + 1$. We then have $\log(a^{-1})^n = n\log(a^{-1}) > \log M$ so $a^{-n} > M$ and $|a^n| = a^n > Ma^{2n} = M|a^{2n}|$. We conclude that $a^n \neq O(a^{2n})$.