Examination 2 Solutions

CS 336

1. [20] Using only Definition 2', prove that the set of infinite strings of 0s and 1s is infinite**.**

Let S = set of infinite strings of 0s and 1s. For $s \in S$, define $f : S \to S$ by $f(s) = 0 \mid s$ (i.e. the infinite string consisting of 0 concatenated with s). For $s, t \in S$, if $s \neq t$, then $f(s) = 0 \mid s \neq 0 \mid t = f(t)$, so f is one-to-one. Let $u = 111...$ the string of all 1s. Since for all $s \in S$ the first element of $f(s)$ is a 0, there is no $s \in S$ such that $f(s) = u$. We have then that f maps S into $S \sim \{u\}$, which is a proper subset of S, and by Definition 2', S is infinite.

2. [20] Prove the set of intervals $\{[a,b] \mid 0 \le a \le b \le 1\}$ is uncountably infinite.

Consider $g : [0,1] \to \{[a,b] \mid 0 \le a \le b \le 1\}$, defined by $g(x) = [0, x]$, for $x \in [0,1]$. If $x \neq y$, then $g(x) = [0, x] \neq [0, y] = g(y)$, so g is one-to-one , and by Theorems 5 and 11, $\{[a,b] \mid 0 \le a \le b \le 1\}$ is uncountably infinite.

3. [20] Let $FP = \{\textit{permutations of}\ \{0,...,n\} \mid n \in \mathbb{N}\}$. Prove that FP is countably infinite.

Since for every $n \in \mathbb{N}$ there are $(n+1)!$ permutations of $\{0,...,n\}$, the number of permutations is finite and FP is the union of a countably infinite collection of finite sets. By Theorem 9, FP is countable. Define $f : \mathbb{N} \to FP$ by $f(n) = \langle 0,1,...,n \rangle$. For natural numbers n and m, if $n \neq m$, then

$$f(n) = \langle 0,1,...,n \rangle \neq \langle 0,1,...,m \rangle = f(m)$$

so f is one-to-one and by Theorem 4, FP is infinite and thus countably infinite.

4. [20] a. By induction prove that $n \geq 1, n^{n-1} \geq n!$.

For $n = 1$ we have $n^{n-1} = 1^0 = 1 \geq 1 = 1!$. If we assume for some $n \geq 1, n^{n-1} \geq n!$, then we conclude:

$$(n+1)^{(n+1)-1} = (n+1)^n = (n+1)(n+1)^{n-1} \geq (n+1)n^{n-1} \geq (n+1)n! = (n+1)!.$$

So by induction, we have $n \geq 1, n^{n-1} \geq n!$ for all $n \geq 1$.

b. Using part a, prove that $n^n \neq O(n!)$. (You may ignore part a if you have another way of proving this and you may use part a even if you weren't able to prove it above.)

Suppose there exist M and N so that for $n \geq N, |n^n| \leq M|n!|$. If we chose $n = \max\{N, \lceil M \rceil + 1)\}$, we have $n \geq N$ and $n > M$

$$|n^n| = n^n = n \cdot n^{n-1} \geq n \cdot n! > M \cdot n! = M|n!|.$$

This is a contradiction, so $n^n \neq O(n!)$.

5. **[20]** Prove that $2^n = o(n!)$. (Hint: $\prod_{i=1}^{n} 2 = \prod_{i=1}^{n} \frac{2}{i} i$)

Given any $\varepsilon > 0$, let $N = \left\lceil \frac{2}{\varepsilon} \right\rceil$. Thus for $n \geq N$, we have $n \geq \frac{2}{\varepsilon}$ and $\varepsilon \geq \frac{2}{n}$, so

$|2^n| = 2^n = \prod_{i=1}^{n} 2 = \prod_{i=1}^{n} \frac{2}{i} i = \prod_{i=1}^{n} \frac{2}{i} \cdot \prod_{i=1}^{n} i \leq \frac{2}{n} \prod_{i=1}^{n} i \leq \varepsilon \prod_{i=1}^{n} i = \varepsilon |i!|$. We conclude $2^n = o(n!)$.

6. **[20]** Prove that for $k \geq 0, \sum_{i=0}^{k} a_i n^i = O(n^k)$.

By Theorem 6, $0 \leq i \leq k, n^i = O(n^k)$. By Theorem 1, $0 \leq i \leq k, a_i n^i = O(n^k)$. By Corollary 3.2, $k \geq 0, \sum_{i=0}^{k} a_i n^i = O(n^k)$.