CS 336

1. The important issue is the logic you used to arrive at your answer.
2. Use extra paper to determine your solutions then neatly transcribe them onto these sheets.
3. Do not submit the scratch sheets. However, all of the logic necessary to obtain the solution should be on these sheets.
4. Comment on all logical flaws and omissions and enclose the
comments in boxes

1. [10] How many five digit decimal numbers (allowing leading zeros) either begin with 3, end with 5, or contain at least one 7 someplace? (**Do not waste time simplifying**.)

We use the principle of inclusion and exclusion. Let $B_3 = \{\text{five digit decimal numbers beginning with a 3}\}$, $E_5 = \{\text{five digit decimal numbers ending with a 5}\}$, and $S_7 = \{\text{five digit decimal numbers containing a 7}\}$. We have $|B_3| = 10^4, |E_5| = 10^4$, $|S_7| = 10^5 - 9^5, |B_3 \cap E_5| = 10^3, |B_3 \cap S_7| = 10^4 - 9^4, |E_5 \cap S_7| = 10^4 - 9^4$, and $|B_3 \cap E_5 \cap S_7| = 10^3 - 9^3$. We conclude

$$\begin{aligned}
&|B_3 \cap E_5 \cap S_7| \\
&= |B_3| + |E_5| + |S_7| - |B_3 \cap E_5| - |B_3 \cap S_7| - |E_5 \cap S_7| + |B_3 \cap E_5 \cap S_7| \\
&= 10^4 + 10^4 + (10^5 - 9^5) - 10^3 - (10^4 - 9^4) - (10^4 - 9^4) + (10^3 - 9^3).
\end{aligned}$$

2. [10] Consider arrays of the form $< r_1, r_2, r_3, r_4 >$, where $r_1 \geq 2, r_2 \geq 3, r_3 \geq 4,$ and $r_4 \geq 5$. How many such arrays are there satisfying

$$r_1 + r_2 + r_3 + r_4 = 25.$$

(Hint: Consider the excesses.)

If we consider the excess variables $n_1 = r_1 - 2, n_2 = r_2 - 3, n_3 = r_3 - 4,$ and $n_4 = r_4 - 5$, the problem is equivalent to "How many arrays $< n_1, n_2, n_3, n_4 >$ of non-negative integers are there satisfying $n_1 + n_2 + n_3 + n_4 = 11$?". If we label four bins n_1, n_2, n_3, n_4, respectively, and place 11 identical balls into the bins, we can do it in $\binom{11+4-1}{3}$ different ways. This is the number of arrays of the form $< r_1, r_2, r_3, r_4 >$, where $r_1 \geq 2, r_2 \geq 3, r_3 \geq 4,$ and $r_4 \geq 5.$ and $r_1 + r_2 + r_3 + r_4 = 25.$

3. a. [10] Using a combinatorial argument, prove that for $n \geq 2$ and $m \geq 2$:

$$\binom{n+m}{2} = n \cdot m + \binom{n}{2} + \binom{m}{2}$$

Let A and B be disjoint sets of cardinalities n and m, respectively. We seek to determine how many subsets of two elements there are in $A \cup B$. Since the cardinality of $A \cup B$ is $n+m$, there are $\binom{n+m}{2}$ such subsets. Alternatively, we could obtain such a subset by selecting one element from each of A and B, by selecting both elements from A, or by selecting both elements from B. There are $nm + \binom{n}{2} + \binom{m}{2}$ ways of doing this and, therefore $\binom{n+m}{2} = nm + \binom{n}{2} + \binom{m}{2}$.

b. [10] Using a combinatorial argument, prove that for $n \geq 1$:

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$$

(Hint: Let A be a set of cardinality *n*. Consider pairs $< a, B >$ where $a \in A \sim B$ and $B \subseteq A \sim \{a\}$.)

Employing the notation from the hint, and considering the left side of the equation first, there are n choices for a and then 2^{n-1} subsets from the remaining $n-1$ elements. Alternatively, let k be the number of elements in $\{a\} \cup B$. The value of k could range from 1 through n. For a fixed value of k, there are $\binom{n}{k}$ ways to choose $\{a\} \cup B$, and then k choices from this for a (with the remaining chosen elements forming B). There are $\sum_{k=1}^{n} k\binom{n}{k}$ total ways of doing this and this must equal $n2^{n-1}$.

4. a. [5] Three dice are rolled. Consider all ordered outcomes equally likely. What is the probability that at least one die shows a 4 given that the sum of the rolls is 12?

There are 25 equally likely cases in which the sum of the rolls is 25: $\langle 6,5,1 \rangle$, $\langle 6,4,2 \rangle$, $\langle 6,3,3 \rangle$, $\langle 6,2,4 \rangle$, $\langle 6,1,5 \rangle$, $\langle 5,6,1 \rangle$, $\langle 5,5,2 \rangle$, $\langle 5,4,3 \rangle$, $\langle 5,3,4 \rangle$, $\langle 5,2,5 \rangle$, $\langle 5,1,6 \rangle$, $\langle 4,6,2 \rangle$, $\langle 4,5,3 \rangle$, $\langle 4,4,4 \rangle$, $\langle 4,3,5 \rangle$, $\langle 4,2,6 \rangle$, $\langle 3,6,3 \rangle$, $\langle 3,5,4 \rangle$, $\langle 3,4,5 \rangle$, $\langle 3,3,6 \rangle$, $\langle 2,6,4 \rangle$, $\langle 2,5,5 \rangle$, $\langle 2,4,6 \rangle$, $\langle 1,6,5 \rangle$, and $\langle 1,5,6 \rangle$. Of these 13 include at least one 4. The probability that at least one die shows a 4 given that the sum of the rolls is 12 is $12/25$.

b. [5] Three dice are rolled. Consider all ordered outcomes equally likely. Is the event that at least one die shows a 4 statistically independent of the event that the sum of the rolls is 12?

There are 6^3 different equally likely outcomes. Of those 5^3 have no 4, thus $6^3 - 5^3$ have at least one 4. Thus the probability of having at least one die showing a 4 is $\frac{6^3 - 5^3}{6^3} = \frac{91}{216}$. Since the probability that least one die shows a 4 given that the sum of the rolls is 12 is $12/25$ and this is not equal to $\frac{91}{216}$, the events are not statistically independent.

5. [10] Prove: The unit circle $C = \{x + iy : x^2 + y^2 = 1\}$ in the complex plane is uncountably infinite.

Consider the function $f : [0,1] \to C$ defined by $f(x) = x + \sqrt{1 - x^2}\, i$. Notice for $x \in [0,1]$, that $0 \le 1 - x^2$. The function f is one-to-one since if $x, y \in [0,1]$ and $x \ne y$, then $f(x) = x + \sqrt{1 - x^2}\, i \ne y + \sqrt{1 - y^2}\, i = f(y)$. Since from Theorem 5, the interval $[0,1]$ is uncountably infinite, Theorem 12 guarantees that C is also uncountably infinite.

6. [10] Let A be a nonempty set. Prove that $\mathcal{P}(A)$, the power set of A, cannot be put into one-to-one correspondence with A (i.e., there exists no function $f : A \xrightarrow[onto]{1-1} \mathcal{P}(A)$). (Hint: What about elements $a \in A$ satisfying $a \notin f(a)$?)

Suppose there exists a function $f : A \xrightarrow[onto]{1-1} \mathcal{P}(A)$. Let $C = \{a \mid a \in A \text{ and } a \notin f(a)\}$ and notice that $C \subseteq A$ so $C \in \mathcal{P}(A)$. Since f is onto there exists some $\bar{a} \in A$ so that $f(\bar{a}) = C$. We must then either have that $\bar{a} \in C$ or $\bar{a} \notin C$. If $\bar{a} \in C$ we have a contradiction since $\bar{a} \in C$ implies that $\bar{a} \notin f(\bar{a}) = C$. But if $\bar{a} \notin C$ we also have a contradiction since in that case $\bar{a} \in f(\bar{a}) = C$. Since both assuming that $\bar{a} \in C$ and $\bar{a} \notin C$ result in contradictions, we conclude that no function $f : A \xrightarrow[onto]{1-1} \mathcal{P}(A)$.

7. [10] Given $f : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}$ prove that if $f = O(g)$ then $f^2 = O(g^2)$ (where $f^2(n) = (f(n))^2$).

By definition, there exist non-negative constants M and N such that for all $n \ge N$, $|f(n)| \le M|g(n)|$. It follows that for all $n \ge N$, $|f^2(n)| = |(f(n))^2| \le M^2 |(g(n))^2| = M^2 |g^2(n)|$ so $f^2 = O(g^2)$.

8. [10] . Prove that for any $f:\mathbb{N}\to\mathbb{R}$, the function $g:\mathbb{N}\to\mathbb{R}$ defined as

$$g(n)=\begin{cases} 1 & n=0 \\ \frac{1}{n}f(n) & n\geq 1 \end{cases}$$ satisfies $g=o(f)$.

Given $\varepsilon>0$, let $N=\max\{1,\lceil 1/\varepsilon\rceil\}$. For $n\geq N$, we have $n\geq 1$ and $n\geq 1/\varepsilon$, so $1/n\leq\varepsilon$ and $|g(n)|=|\frac{1}{n}f(n)|\leq\varepsilon|f(n)|$. We conclude $g=o(f)$.

9. [10] Assuming x and y are integer variables, prove correct with respect to precondition "y is defined" and postcondition "$x \geq 1$":

```
if y > 0 then
        x := y+6
        if x > 11 then
                x : = x-10
        endif
else
        x := 4-y
        y := y-1
        if y = -3 then
                x : = x-3
        endif
endif
```

Code	Assertion
	y is defined
if y > 0 **then**	$y > 0$
x := y+6	$y > 0 \wedge x = y + 6$
	$x > 6$
if x > 11 **then**	$x > 11$
x : = x-10	$(x' > 11) \wedge (x = x' - 10)$
	$x > 1$
endif	$(x > 6) \vee (x > 1)$
	$x > 1$
else	$y \leq 0$
x := 4-y	$(y \leq 0) \wedge (x = 4 - y)$
	$x \geq 4$
y := y-1	$x \geq 4$
if y = -3 **then**	$x \geq 4$
x : = x-3	$(x' \geq 4) \wedge (x = x' - 3)$
	$x \geq 1$
endif	$(x \geq 4) \vee (x \geq 1)$
	$x \geq 1$
endif	$(x > 1) \vee (x \geq 1)$
	$x \geq 1$

10. [10] Consider a function $parity : \mathbb{N} \to \{0,1\}$ defined by $parity(n) = \begin{cases} 0 & if\, n\, is\, even \\ 1 & if\, n\, is\, odd \end{cases}$. Prove the following code is partially correct with respect to precondition "$n \geq 0$" and postcondition "$p = parity(n)$". (Assume p and i are integer variables.)

```
p := 0
i := 1
while i ≤= n do
        p := 1-p
        i := i+1
endwhile
```

Be explicit about your loop invariant: $\mathrm{I} = ((p = parity(i-1)) \wedge (i \leq n+1))$

(Hint: You may want to prove a lemma: $\forall n \in \mathbb{N}, parity(n) = 1 - parity(n-1).$)

Lemma: $\forall n \in \mathbb{N}, parity(n) = 1 - parity(n-1).$

Proof:

$\forall n \in \mathbb{N}, even(n) \Rightarrow (parity(n) = 0) \wedge (odd(n-1))$

$\Rightarrow (parity(n) = 0) \wedge (parity(n-1) = 1)$

$\Rightarrow (parity(n) = 1 - parity(n-1))$

$\forall n \in \mathbb{N}, odd(n) \Rightarrow (parity(n) = 1) \wedge (even(n-1))$

$\Rightarrow (parity(n) = 1) \wedge (parity(n-1) = 0)$

$\Rightarrow (parity(n) = 1 - parity(n-1))$

____________________________ $n \geq 0$

p := 0_______________________ $(n \geq 0) \wedge (p = 0)$

i := 1_______________________ $(n \geq 0) \wedge (p = 0) \wedge (i = 1)$

____________________ $(p = parity(i-1)) \wedge (i \leq n+1)$

while i ≤= n **do**_____ $(p = parity(i-1)) \wedge (i \leq n+1) \wedge (i \leq n)$

________ $(p = parity(i-1)) \wedge (i \leq n)$

p := 1-p______ $(p' = parity(i-1)) \wedge (i \leq n) \wedge (p = 1 - p')$

________ $(p = parity(i)) \wedge (i \leq n)$

i := i+1_______ $(p = parity(i')) \wedge (i' \leq n) \wedge (i = i'+1)$

________ $(p = parity(i-1)) \wedge (i \leq n+1)$

endwhile_________________ $(p = parity(i-1)) \wedge (i \leq n+1) \wedge (i > n)$

___________________________ $(p = parity(i+1)) \wedge (i = n+1)$

___________________________ $p = parity(n)$

11. [10] Prove that the code below terminates. (Assume s and i are integer variables.):

```
s := 0
i := 1
while i ≤= 100000 do
        s := s+i
        i := 4*i+2
endwhile
```

First we recognize that if the quantity *100,000-i* becomes negative, the loop will terminat. We will show that that quantity strictly decreases but to that end we need to guarantee that the variable i stays positive. Consider the invariant "$i \geq 1$":

```
s := 0
i := 1____________________ i ≥ 1
while i ≤= 100000 do_____ i ≥ 1
        s := s+i ____________ i ≥ 1
        i := 4*i+2___________ (i' ≥ 1) ∧ (i = 4i' + 2)
             _______________ i = i' + 3i' + 2 > i' ≥ 1
             _______________ (i ≥ 1) ∧ (100000 − i < 100000 − i')
             _______________ i ≥ 1
endwhile
```

The quantity *100,000-i* strictly decreases through the loop. Since this is an integer expression, eventually *100,000-i* becomes negative and the loop terminates.

12. [10] Determine the weakest precondition with respect to the postcondition "$S = 0$" for the following (assume S, y, and x are integer variables and y and x are defined

```
if x≠0 then
        x := y
        S := x-y
else
        S := y+x
endif
```

wp(**if** x≠0 **then** x := y; S := x-y **else** S := y+x **endif,** $S = 0$)
$= ((x \neq 0) \wedge wp($ x := y; S := x-y, $S = 0)) \vee ((x = 0) \wedge wp($ S := y+x, $S = 0))$
$= ((x \neq 0) \wedge wp($ x := y; $x - y = 0)) \vee ((x = 0) \wedge (y + x = 0))$
$= ((x \neq 0) \wedge (y - y = 0)) \vee (x = y = 0))$
$= (x \neq 0) \vee (x = y = 0))$

13. [10] Determine the weakest precondition with respect to the postcondition "$x \neq y$" for the following (assume y and x are integer variables and x is defined). Simplify your answer so that there are **NO** logical operators.

```
if x ≥ 3 then
        y:= 2
else
        if x = 2 then
                y := 6
        else
                y := x+1
        endif
endif
```

We consider the inner if-then-else first:

wp (**if** $x = 2$ **then** y := 6 **else** y := x+1 **endif,** $x \neq y$)

$= ((x = 2) \wedge wp(\text{y := 6}, \ x \neq y)) \vee ((x \neq 2) \wedge wp(\text{ y := x+1}, \ x \neq y))$

$= ((x = 2) \wedge (x \neq 6)) \vee ((x \neq 2) \wedge (x \neq x + 1))$

$= ((x = 2) \vee (x \neq 2)$

$= true$

Now, letting S denote "**if** $x = 2$ **then** y := 6 **else** y := x+1 **endif**",

wp (**if** $x \geq 3$ **then** y:= 2 **else** S **endif,** $x \neq y$)

$= ((x \geq 3) \wedge wp(\text{y := 2}, \ x \neq y)) \vee ((x < 3) \wedge wp(S, \ x \neq y))$

$= ((x \geq 3) \wedge (x \neq 2)) \vee ((x < 3) \wedge true)$

$= ((x \geq 3) \vee (x < 3)$

$= true$

So, wp (**if** $x \geq 3$ **then** y:= 2 **else if** $x = 2$ **then** y := 6 **else** y := x+1 **endif endif,** $x \neq y$) = $true$.