CS 336
Final Examination Solutions

1. a. [5] Given $|A_1| = n_1, |A_2| = n_2, |A_3| = n_3,$ and $|B| = m,$ how many functions map $A_1 \times A_2 \times A_3$ into B?

The domain of such a function is a set of cardinality $n_1 \cdot n_2 \cdot n_3$. For a given function, there are m options for the value defined for each variable – thus there are $m^{n_1 n_2 n_3}$ such functions.

b. [5] How many such functions mapping $A_1 \times A_2 \times A_3$ into B are one-to-one?

If $m < n_1 \cdot n_2 \cdot n_3$ there are no one-to-one functions mapping $A_1 \times A_2 \times A_3$ into B **.** If $m \geq n_1 \cdot n_2 \cdot n_3$, for a given function, there are m options for the value defined for a first variable, $m-1$ options for the value defined for a second variable, …, and $m - n_1 n_2 n_3 + 1$ options for the value defined for a $n_1 n_2 n_3$ th variable – thus there are $m \cdot (m-1) \cdot \cdots \cdot (m - n_1 n_2 n_3 + 1)$ such one-to-one functions.

2. a. [10] How many triples $\langle i, j, k \rangle$ of non-negative values $i, j,$ and k satisfy

$$i + j + k \leq 20?$$

(Hint: Consider a variable $l = 20 - (i + j + k)$.)

Consider placing 20 indistinguishable balls into four bins labeled $i, j, k,$ and l. Since the number of balls in the l bin is non-negative, each such placement corresponds to a selection of a four-tuple $\langle i, j, k, l \rangle$ of non-negative values $i, j, k,$ and l satisfying $i + j + k + l = 20$. But since $l \geq 0$, this is equivalent to a selection of a triple $\langle i, j, k \rangle$ of non-negative values $i, j,$ and k satisfying $i + j + k \leq 20$. There are $\binom{20+3}{3}$ such placements of 20 indistinguishable balls into four bins, therefore the same number of triples $\langle i, j, k \rangle$ of non-negative values $i, j,$ and k satisfying $i + j + k \leq 20$.

b. [5] How many triples $\langle i, j, k\rangle$ of non-negative values $i, j,$ and k satisfy

$$4 \le i + j + k \le 20\,?$$

For $n = 0,1,2,3,$ there are $\binom{n+2}{2}$ such placements of n indistinguishable balls into three bins, so there are $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\binom{5}{2}$ triples $\langle i, j, k\rangle$ of non-negative values $i, j,$ and k satisfying $i + j + k < 4$. Since there are $\binom{23}{3}$ triples $\langle i, j, k\rangle$ of non-negative values $i, j,$ and k satisfying $i + j + k \le 20$, there are $\binom{23}{3}-\binom{2}{2}-\binom{3}{2}-\binom{4}{2}-\binom{5}{2}$ triples $\langle i, j, k\rangle$ of non-negative values $i, j,$ and k satisfying $4 \le i + j + k \le 20$.

3. a. [10] Using a combinatorial argument, prove that for $n \geq m \geq 3$,:

$$\binom{n}{m} = \binom{n-3}{m} + \binom{n-3}{m-1}\binom{3}{1} + \binom{n-3}{m-2}\binom{3}{2} + \binom{n-3}{m-3}$$

Let A be a set of cardinality $n-3$, B be a disjoint set of cardinality 3, and $C = A \cup B$. Since A and B are disjoint, C has cardinality n. Consider the number of subsets of C of cardinality $m \leq n$. We select without order and without repetitions so there are $\binom{n}{m}$ such subsets. Alternatively, there are four options corresponding to the number of the elements in the subset that come from B: zero, one, two, or three. If there are zero elements in the subset that come from B, then all elements come from A, so there are $\binom{n-3}{m}$ such subsets. If there is one element in the subset that comes from B, then there are $m-1$ elements that come from A, so there are $\binom{n-3}{m-1}$ ways to choose the elements from A and $\binom{3}{1}$ ways to choose the single element from B. If there are two elements in the subset that come from B, then there are $m-2$ elements that come from A, so there are $\binom{n-3}{m-2}$ ways to choose the elements from A and $\binom{3}{2}$ ways to choose the two elements from B. Finally, if there are three elements in the subset that come from B, then there are $m-3$ elements that come from A, so there are $\binom{n-3}{m-3}$ ways to choose the elements from A and one way to choose the three elements from B. Together then there are $\binom{n-3}{m} + \binom{n-3}{m-1}\binom{3}{1} + \binom{n-3}{m-2}\binom{3}{2} + \binom{n-3}{m-3}$ subsets and this must equal $\binom{n}{m}$.

b. [10] Using a combinatorial argument, prove that for $m, n, p \geq 0$:

$$\sum_{k=0}^{n} \binom{n}{k} m^k p^{n-k} = (m+p)^n$$

Let A be a set of cardinality m, B be a disjoint set of cardinality p, and $C = A \cup B$. Since A and B are disjoint, C has cardinality $m + p$. Consider the number of strings of length n selected from the integers C. For the right side, we see there are n positions in the string each of which has $m + p$ options. We select with order and allow repetition so there are $(m+p)^n$ such strings. Alternatively, let k indicate the number of positions in the string occupied by an element of A. The value of k can vary from 0 to n. For a fixed value of k, there are $\binom{n}{k}$ ways to choose the positions for those elements and, with the positions fixed, there are m choices for each of the k positions. Finally, the elements of B must occupy the remaining $n-k$ positions of the string and there are p choices for each of the $n-k$ positions. Thus, there $\binom{n}{k} m^k p^{n-k}$ such strings for a fixed value of k and $\sum_{k=0}^{n} \binom{n}{k} m^k p^{n-k}$ strings overall. This must equal $(m+p)^n$.

4. a. [10] For $n \geq 5$, what is the probability that a string of *n* zeros and ones has exactly 5 ones. (You may assume all strings of *n* zeros and ones are equally probable.)

There are 2^n equally likely strings of zeros and ones of length n. Of these there are $\binom{n}{5}$ that have exactly 5 ones. Thus, probability that a string of *n* zeros and ones has exactly 5 ones is $\frac{\binom{n}{5}}{2^n}$.

b. [5] For $n \geq 5$, what is the probability that a string of *n* zeros and ones has exactly 5 ones given that it has at least three ones. (You may assume all strings of *n* zeros and ones are equally probable.)

There is one string with no 1s, there are n strings with one 1, and there are $\binom{n}{2}$ strings with two 1s, so there are $2^n - 1 - n - \binom{n}{2}$ equally likely that have at least three ones. All strings with five 1s have at least three 1s, so the number of strings with five one and at least three ones is $\binom{n}{2}$. Thus, the probability that a string of *n* zeros and ones has exactly 5 ones given that it has at least three ones is

$$\frac{\binom{n}{2}}{2^n - 1 - n - \binom{n}{2}}.$$

5. **[10]** Using definition 2' (and no cardinality theorems) prove that $\mathrm{N} \times \mathrm{N}$, the set of ordered pairs of natural numbers, is infinite.

Consider the mapping $f : \mathrm{N} \times \mathrm{N} \to \mathrm{N} \times \mathrm{N}$ defined by $f(i, j) = (i+1, j)$. If $(i, j) \in \mathrm{N} \times \mathrm{N}$ then $(i+1, j) \in \mathrm{N} \times \mathrm{N}$. For $(i_1, j_1) \in \mathrm{N} \times \mathrm{N}$ and $(i_2, j_2) \in \mathrm{N} \times \mathrm{N}$ with $(i_1, j_1) \neq (i_2, j_2)$ then either $i_1 \neq i_2$ or $j_1 \neq j_2$ and thus either $i_1 + 1 \neq i_2 + 1$ or $j_1 \neq j_2$. In either case $f(i_1, j_1) = (i_1 + 1, j_1) \neq (i_2 + 1, j_2) = f(i_2, j_2)$, so f is one-to-one. However, for no element $(i, j) \in \mathrm{N} \times \mathrm{N}$ is $f(i, j) = (0, 0)$ since that would imply that $i = -1$. We conclude that f maps $\mathrm{N} \times \mathrm{N}$ one-to-one into a proper subsets of itself, and thus is infinite.

6. [10] Prove this corollary to Theorem 10:

Given a countably infinite collection of finite sets $\{A_i\}_{i\in\mathbb{N}}$ *satisfying* $A_0 \neq \varnothing$ *and for* $i \geq 1$,

$$A_i \not\subset \bigcup_{j=0}^{i-1} A_j$$

the union $\bigcup_{i\in\mathbb{N}} A_i$ *is countably infinite. (In other words, if each set contains at least one element not contained in its predecessors, the union cannot be finite.)*

Theorem 10 guarantees that $\bigcup_{i\in\mathbb{N}} A_i$ is countable. For each $i \in \mathbb{N}$, select $a_i \in A_i \sim \bigcup_{j=0}^{i-1} A_j$. Define $f : \mathbb{N} \to \bigcup_{i\in\mathbb{N}} A_i$ by $f(i) = a_i$. For $i_1 \neq i_2$, assume without loss of generality that $i_1 < i_2$, then $f(i_1) = a_{i_1} \in A_{i_1} \subseteq \bigcup_{j=0}^{i_2-1} A_j$ but $f(i_2) = a_{i_2} \notin \bigcup_{j=0}^{i_2-1} A_j$, so $f(i_1) \neq f(i_2)$ and f is one-to-one. By Theorem 4, $\bigcup_{i\in\mathbb{N}} A_i$ is infinite and thus countably infinite.

7. [10] Prove that for $a > 1$ and $0 < \lambda < 1$, $a^n \neq \mathrm{O}(a^{\lambda n})$.

Given any non-negative constants M and N . Notice that $M + 1 > 0$ and $0 < 1 - \lambda < 1$. Take $n = \left\lceil \max\{N, \frac{\log(M+1)}{(1-\lambda)\log(a)}\right\rceil$. We have $n \geq N$ and $n \geq \frac{\log(M+1)}{(1-\lambda)\log(a)}$, so $n \cdot (1-\lambda)\log(a) \geq \log(M+1)$ and $a^{(1-\lambda)n} \geq M + 1 > M$. So $|a^n| = a^n = a^{(1-\lambda)n} \cdot a^{\lambda n} > Ma^{\lambda n} = M\,|a^{\lambda n}|$ and $a^n \neq \mathrm{O}(a^{\lambda n})$.

8. [10] . Prove **Theorem 7:** If $a < b$, then $n^a = o(n^b)$

Given any $\varepsilon > 0$, let $N = (1/\varepsilon)^{1/(b-a)}$. Notice then for $n \geq N = (1/\varepsilon)^{1/(b-a)}$, $n^{b-a} \geq 1/\varepsilon$, and $n^{-(b-a)} \leq \varepsilon$. So $\left|n^a\right| = \left|n^{-(b-a)}\ n^b\right| = \left|n^{-(b-a)}\right|\left|n^b\right| \leq \varepsilon\ \left|n^b\right|$. Therefore, $n^a = o(n^b)$.

9. [10] Assuming x and y are integer variables, prove correct with respect to precondition "y is defined" and postcondition "$x \neq y$":

```
if y > 3 then
        x := y+6
        if x < 11 then
                y : = 11
        endif
else
        x := y-2
        y := y-1
endif
```

______________________	y *is defined*
if y > 3 **then** __________	$y > 3$
x := y+6	
________________	$(y > 3) \wedge (x = y + 6)$
________________	$x \neq y$
if x < 11 **then**____	$(x \neq y) \wedge (x < 11)$
__________	$x < 11$
y : = 11 ___	$(y = 11) \wedge (x < 11)$
__________	$x \neq y$
endif___________	$(x \neq y) \vee (x \neq y)$
________________	$x \neq y$
else _________________	$y \leq 3$
x := y-2_________	$(y \leq 3) \wedge (x = y - 2)$
________________	$x = y - 2$
y := y-1_________	$(y = y' - 1) \wedge (x = y' - 2)$
________________	$x = y - 1$
________________	$x \neq y$
endif_________________	$(x \neq y) \vee (x \neq y)$
______________________	$x \neq y$

10. [10] Prove the following code is partially correct with respect to precondition "true" and postcondition " $x = 1$ " (assume x is an integer variable.):

```
x := 0
while x = 0 do
        x := 1
end
```

Be explicit about your loop invariant: I =

Use I $= (x \neq 0) \Rightarrow (x = 1)$

______________________________	*true*
x := 0______________________	$x = 0$
______________________________	$(x \neq 0) \Rightarrow (x = 1)$
while x = 0 **do**______________	$(x = 0) \wedge ((x \neq 0) \Rightarrow (x = 1))$
x := 1________________	$x = 1$
______________________	$(x \neq 0) \Rightarrow (x = 1)$
end____________________	$(x \neq 0) \wedge ((x \neq 0) \Rightarrow (x = 1))$
______________________________	$x = 1$

11. [10] Prove partial correctness with respect to precondition $n \geq 1$ and postcondition $p = m^n$ (assume *m, p, i,* and *n* are integer variables and that *m* is defined.):

```
p :=1
i := 1
while i ≤ n do
        p := m*p
        i := i+1
end
```

Be explicit about your loop invariant: I =

Loop invariant: $\mathrm{I} = ((p = m^{i-1}) \wedge (i \leq n+1))$

______________	$n \geq 1$
p :=1 ______________	$p = 1 \wedge n \geq 1$
i := 1 ______________	$i = 1 \wedge p = 1 \wedge n \geq 1$
______________	$p = m^{i-1} \wedge i \leq n+1$
while $i \leq n$ **do** ______________	$i \leq n \wedge p = m^{i-1} \wedge i \leq n+1$
______________	$i \leq n \wedge p = m^{i-1}$
p := m*p ______________	$p = mp' \wedge i \leq n \wedge p' = m^{i-1}$
______________	$p = m^{i} \wedge i \leq n$
i := i+1 ______________	$i = i' + 1 \wedge p = m^{i'} \wedge i' \leq n$
______________	$p = m^{i-1} \wedge i \leq n+1$
end ______________	$i > n \wedge p = m^{i-1} \wedge i \leq n+1$
______________	$i = n+1 \wedge p = m^{i-1}$
______________	$p = m^{n}$

12. a [5] Assuming min, b, and c are integer variable and that b, and c are defined, determine the weakest precondition with respect to the postcondition

"$(\min = a \vee \min = b \vee \min = c) \wedge (\min \leq a) \wedge (\min \leq b) \wedge (\min \leq c)$":

if b < c **then**
    min : = b
else
    min := c
endif

$$
\begin{aligned}
& wp(\min := \mathrm{c},\ (\min = a \vee \min = b \vee \min = c) \wedge (\min \leq a) \wedge (\min \leq b) \wedge (\min \leq c)) \\
& \qquad = (c = a \vee c = b \vee c = c) \wedge (c \leq a) \wedge (c \leq b) \wedge (c \leq c) \\
& \qquad = true \wedge (c \leq a) \wedge (c \leq b) \wedge true \\
& \qquad = (c \leq a) \wedge (c \leq b) \\
& wp(\min := \mathrm{b},\ (\min = a \vee \min = b \vee \min = c) \wedge (\min \leq a) \wedge (\min \leq b) \wedge (\min \leq c)) \\
& \qquad = (b = a \vee b = b \vee b = c) \wedge (b \leq a) \wedge (b \leq b) \wedge (b \leq c) \\
& \qquad = true \wedge (c \leq a) \wedge true \wedge (b \leq c) \\
& \qquad = (b \leq a) \wedge (b \leq c) \\
& wp(\textbf{if}\ \mathrm{b} < \mathrm{c}\ \textbf{then}\ \min := \mathrm{b}\ \textbf{else}\ \min := \mathrm{c}\ \textbf{endif}, \\
& \qquad (\min = a \vee \min = b \vee \min = c) \wedge (\min \leq a) \wedge (\min \leq b) \wedge (\min \leq c)) \\
& = ((b < c) \wedge (b \leq a) \wedge (b \leq c)) \vee ((b \geq c) \wedge (c \leq a) \wedge (c \leq b)) \\
& = ((b < c) \wedge (b \leq a)) \vee ((c \leq b) \wedge (c \leq a))
\end{aligned}
$$

b. [5] Letting pre represent the precondition found in a, prove that $(b < a) \Rightarrow pre$.

We have $pre = ((b < c) \wedge (b \leq a)) \vee ((c \leq b) \wedge (c \leq a))$. If $b < a$ then either $(b < a) \wedge (b < c)$ or $(b < a) \wedge (b \geq c)$. If $(b < a) \wedge (b < c)$ then certainly $(b < c) \wedge (b \leq a)$. If $(b < a) \wedge (b \geq c)$ then $c \leq b < a$ and certainly $(c \leq b) \wedge (c \leq a)$. Thus, $(b < a) \Rightarrow pre$.

13. [10] Determine the weakest precondition with respect to the postcondition "$z = 2$" for the following (assume z, y, and x are integer variables). Simplify your answer so that there are NO logical operators.

x := 3
z := 2*x-y
if *y>0* **then**
    z := z-2
else
    z := -z
endif

$$wp(\text{z := z - 2}, z = 2) = (z - 2 = 2) = (z = 4)$$
$$wp(\text{z := -z}, z = 2) = (-z = 2) = (z = -2)$$
So $wp(\textbf{if } \text{y>0 } \textbf{then } \text{z := z-2 } \textbf{else } \text{z := -z}, z = 2) = (((\text{ y>0}) \wedge (\text{z=4})) \vee ((\text{ y} \leq 0) \wedge (\text{z=-2})))$
$$wp(\text{z := 2* x - y}, = (((\text{ y>0}) \wedge (\text{2x-y=4})) \vee ((\text{ y} \leq 0) \wedge (\text{2x-y=-2})))$$
$$= (((\text{ y>0}) \wedge (\text{2x=4+y})) \vee ((\text{ y} \leq 0) \wedge (\text{2x=y-2})))$$
$$wp(\text{x := 3}, ((\text{ y>0}) \wedge (\text{2x=4+y})) \vee ((\text{ y} \leq 0) \wedge (\text{2x=y-2})))$$
$$= (((\text{ y>0}) \wedge (2 \cdot 3\text{=4+y})) \vee ((\text{ y} \leq 0) \wedge (2 \cdot 3\text{=y-2})))$$
$$= (((\text{ y>0}) \wedge (\text{2=y})) \vee ((\text{ y} \leq 0) \wedge (\text{8=y})))$$
$$= ((\text{2=y}) \vee \mathit{false})$$
$$= (\text{2=y})$$